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Abstract. I study a pure exchange economy with many agents whose marginal utility of consumption is stochastically heterogeneous and independent of wealth, with realized marginal utility of wealth private information. I show that currency and credit are equivalent to a contract that is efficient when the privacy of information is viewed as a constraint, and that the cash in advance constraint is an expression of an incentive constraint.

1. INTRODUCTION

It is a prevailing tradition in macroeconomics that money and credit are distinct institutions. By this tradition, in a purely monetary economy, some individuals will be liquidity-constrained—that is, they won't always have enough money on hand to make purchases even though their marginal utility of consumption is high. It would make sense to form coalitions to overcome the presumably unnecessary restrictions imposed by money so the liquidity-constrained individuals could consume. The tradition seems to be that credit markets provide such a superior arrangement. [1.1]

Green (1987) showed that when information about states is private, credit markets are in fact imperfect in that they do not fully solve the problem; indeed, the use of credit markets for consumption-smoothing can be taken as a symptom of the existence of informational asymmetries. The point I wish to make here, at least within the technical confines of the model I use, is that credit markets are no better than currency markets at overcoming liquidity frictions when there is private information. Money is in fact not a second rate asset.

I demonstrate this equivalence with two modifications of Green's contract-building approach. The first modification is to relax the assumption that the principal and agent discount the future at the same rate. The second modification is to require aggregate resource feasibility. I find that a resource-feasible contract that attains the efficiency imputed to the existence of credit is equivalent to both credit and currency equilibria. Not only are currency and credit equivalent, then, but both can attain the efficiency frontier when all the constraints facing an economy are accounted for; those constraints must express the fact that information is private. The contract approach highlights the fact that currency and credit equilibria are insurance mechanisms, and that restrictions on asset holdings—like credit constraints and cash in advance constraints—exist as means of eliciting information necessary for the provision of the insurance.

This finding could not have arisen from a representative agent model. In representative agent models of assets, one is forced to impose motives for holding credit and currency since assets are unnecessary. In currency equilibria, one is forced to motivate the demand for currency with cash-in-advance constraints or by putting money in the utility function. In credit equilibria, it is necessary to impose borrowing restrictions, typically by imposing transversality conditions on debt holding as a constraint, not as an endogenous boundary condition. In heterogeneous-agent models with complete information, money and credit are unnecessary because complete insurance is feasible, and complete insurance eliminates all need for asset exchanges. In a completely insured heterogeneous-agent model, as in a representative agent model, no assets are ever actually exchanged, even though their prices can be calculated. But combining privacy of information with heterogeneity provides both a reason for the existence of money and credit—insurance—and endogenously motivates the institutional form the insurance takes—cash in advance constraints and borrowing constraints.

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[1.1] The following quotes should not be interpreted as bills of attainder but as well articulated statements of this view. "In any period, there will be some households in a run of low θ's, with large real balances accumulated but no particular urgency to spend them. There will be others in a run of high θ's, with balances of y and a high marginal utility of current consumption. Here, then, are two sides to a nonexistent credit market on which some would gladly lend at positive interest. ... Can this gap be filled by a government-engineered deflation? ... Clearly not. ... There is a sense in which money is a second rate asset." [Lucas 1980, p. 219]
A similar view was earlier expressed by Frank Hahn: "The necessary conditions for Pareto-efficiency in a world of uncertainty with intertemporal choice will in general be fulfilled by a market economy only if money plays no role. There are therefore no grounds for supposing that the Friedman rule is either necessary or sufficient for Pareto-efficiency since it is of the essence of an explanation for the existence of money that other conventional necessary conditions are violated." [Hahn, 1971. p. 70]

Because currency and credit are insurance mechanisms, the intuition that has arisen from studying insurance with private information can be applied. The insurance within my particular model is incomplete, except in special cases I characterize. The efficiency frontier attainable by currency and credit equilibria therefore also reflects incomplete insurance. As with static insurance with private information, it is also the case that an equilibrium might not even exist; there may be adverse selection. It is not just the privacy of information that causes this adverse selection, but the joint effect of private information and resource feasibility.

I obtain these findings using a modified version of Lucas's (1980) model in which the desire for insurance arises from taste shocks rather than income shocks as in Green's approach. The taste shocks are stochastic and are the marginal utility of consumption in each period. I think of this as corresponding to actual hunger or need for services like medical care that arise randomly. Although in the simple setting here the framework is not very realistic looking, it has the advantage of relative technical transparency, so that the intuition of the findings is accessible, and it can potentially be expanded to be more realistic. Moreover, it forms a new class of dynamic programming problems in which the value function has a closed form solution. Although the boundedness assumption common in abstract infinite horizon models is not met here, I show that the dynamic programming approach of such models carries over. Feasibility and incentive compatibility can be imposed in a way that even has geometric intuition, because wealth effects are intrinsically decoupled from substitution effects.

In order to show that money and credit act like an optimal contract, I must first produce an optimal contract. I do not solve the problem of a planner maximizing the welfare of a collection of agents; instead I solve the problem of a principal maximizing profit subject to the constraints of incentive compatibility. The principal's strategy will be to maximize discounted profit, and that profit may be positive or negative depending on the discount factor available to him. This is the same strategy used by Green, except Green assumed equality between the discount factor of individuals and that of the principal.

A second condition can then be imposed: zero profit. Rather than impose zero profit as a constraint, I ask what parameter values ensure zero profit. Having zero profit is equivalent to having a mechanism that uses exactly the resources available to the economy, a prerequisite of efficient mechanisms.

The zero profit condition is equivalent to having competitive firms provide insurance in the manner of Rothschild and Stiglitz (1976). However, there are many equilibria that are both incentive compatible and have zero profit. The optimal equilibrium is that whose parameter values maximize the welfare—ex ante expected discounted utility—of the representative agent. Whether this optimal equilibrium is the optimal mechanism as well must be shown by a uniqueness argument. I show this by asking what happens if the optimal equilibrium is transcended by decreasing a parameter—the discount factor of the principal—below its threshold, finding that a feasible equilibrium cannot exist there. By also showing that any mechanism that is feasible and incentive compatible cannot surpass the welfare of the corresponding zero-profit equilibrium, I have shown optimality.

The behavior of consumption at this optimum is different from that in suboptimal equilibria: it is nonstationary, a property that emerges in other models as well: Green (1987), Atkeson and Lucas (1992), Thomas and Worrall (1990), and Taub (1990) (although stationary optimal equilibria can exist if the stochastic process is discrete). Why then be concerned with stationary, suboptimal equilibria? If one more constraint is imposed, the class of equilibria shrinks further, and the equilibria in this class are stationary.

The additional constraint is that no individual is ever tempted to defect from the equilibrium and permanently revert to autarky. By its nature an insurance contract that is ex-ante desirable to all participants will be ex post undesirable to some. Even if there is no adverse selection, dynamic contracts are subject to attack from the participants. If participants are free to defect from a contract, under what conditions will the contract nevertheless be immune from such defection? This question is of central importance in dynamic insurance contracts because the redistributive nature of the contract depends on all participants continually contributing their contractual shares. Viewing asset equilibria as expressions of such contracts, the quantity constraints that are essential elements of the equilibrium—borrowing constraints and cash in advance constraints—are not supported by prices. There must be some other force that decentralizes these quantity constraints, and the threat of defection provides this force.

I consider permanent reversion to autarky because it would be the result of a grim trigger strategy in a principal versus agent game, and as such defines the maximal set of equilibria that are immune to punishment strategies. The surviving equilibria obey a folk theorem, but one in which the discount factor of the agents must decrease, rather than rise, to maintain cooperation. The result is that equilibria that obey the three properties of incentive compatibility, feasibility, and immunity from defection are a strict subset of those that obey only the first two properties, and these equilibria are stationary.

This last fact in my view justifies studying asset equilibria that are inefficient. The borrowing constraints must somehow be sustained without the equivalent of a principal, and with noncooperative play by the agents. Defection translates into reneging in an asset equilibrium; the folk theorem states that equilibria exist that prevent such reneging. I think of this as a kind of dynamic consistency. The folk theorem implies that this dynamic consistency is possible only with inefficiently high inflation rates and inefficiently low real interest rates. But this is realistic.

In the following section I set out an intuitive framework to motivate the stochastic linear utility framework and also to motivate how restrictions on asset holdings are the expression of a revelation mechanism. In subsequent sections I show how a contract would work in this setting. The next section explores the bounds on efficiency. Then I show that the contract is equivalent to the credit and currency equilibria, and that the efficiency bounds apply there as well. In the penultimate sections I present folk theorems that interpret the results in terms of adverse selection and a defection concept. There are seven short appendices. The first two work out facts about the principal's problem. The next shows that the solution of the principal's dynamic programming recursion, which is affine in the state and hence easy to characterize, is also the value function. The next repeats the process for the agents in a credit equilibrium. The next to last appendix shows that nothing has been lost by examining only stationary equilibria in this setting. The final one shows that the efficiency frontier has a nonstationary solution, but that this solution is not pathological in other senses.

2. A BABYSITTING COOPERATIVE

While the equivalence between insurance contracts and currency and credit equilibria emerges from technical considerations, I find it helpful to think about the operation of a babysitting cooperative for intuitive purposes. [2.1] A number of families with young children live in a new neighborhood with no teenagers, and far from grandparents. They obtain babysitting by babysitting for each other's children. Their babysitting needs are not always simultaneous, and have a random element due to a need for doctor visits, unplanned shopping trips, car repairs, social invitations, and so on. The random element makes the problem facing the group one of providing insurance. An insurance contract to provide a scheme for allocating babysitting time would need to weigh the importance of a family needing a sitter for a medical visit versus another family's desire for a casual restaurant meal or tennis lessons. A contract that provided babysitting to families whenever a family desired it would quickly degenerate—all families would ask for unlimited babysitting. With those same families providing babysitting, the scheme would be infeasible.

A better scheme would make families reveal the urgency of their need so as to allow triage—doctor visits would take precedence over tennis lessons. In an anonymous atmosphere with purely private information about need, there would be an incentive for families to claim urgent need for babysitting even if this were not the case. The contract design problem imposes incentive constraints that force families to reveal their true need.

The contract can do this by tracking each family's history of babysitting requests and services, imposing limits based on the history. Families with histories of excessive demand for babysitting would be temporarily denied babysitting. The fear that this denial would come at a time of great need would prompt families to refrain from demanding babysitting for frivolous reasons—in other words, triage would be possible. Moreover, if babysitting were rewarded with a relaxation of the denial, there would be an incentive to babysit for other families.

Credit might accomplish the same thing. Families needing babysitting would spend credit balances on other members of the cooperative in return for babysitting, and interest would be paid on unused credit. Some

[2.1] I speak from experience.

restriction would have to prevent a family's borrowing without limit, however, and this restriction would serve as an incentive to conserve on the use of the cooperative's services for fear of having the restriction bind at a moment of urgent need.

Another way to implement the contract would be to issue tickets to co-op members, each ticket worth a fixed amount of babysitting time. The tickets would be earned from other members of the co-op by babysitting for them and expended on the members to obtain babysitting. The essential feature of the ticket system is that a family can run out of tickets. A family bereft of tickets could not then obtain sitting services even if their need was great. The fear of running out would induce families to conserve tickets for nonfrivolous uses and each family would produce its share of babysitting. [2.2]

The ticket system has the basic features of a currency. The requirement that babysitting be purchased with tickets is an implementation of a cash-in-advance constraint. The cash in advance constraints can be viewed as the expression of an incentive constraint, and I show that this is in fact the case in section 10.

In order to show that ticket and credit systems reflect the operation of efficient contracts, I must develop a more abstract notion of contract. A contract must induce families to reveal their states, and it must be feasible, and not waste resources. One way to go at this is to think of each family having access to a private intermediary—a grandparent perhaps—who provides babysitting in return for later repayment in time from the parents through a prearranged contract. If that grandparent makes zero profit, it is a simple extension of logic that the cooperative could use the grandparent's contract rules for its own yielding a feasible and incentive compatible contract. For this reason most of the analytical effort that follows will be devoted to analyzing the one-family grandparents' contract.

I next outline the quantitative features of the families in the cooperative.

3. STOCHASTIC LINEAR UTILITY AND THE ECONOMIC ENVIRONMENT

In this section I describe the central element of the technical models that follow: families with stochastic utility. Each family has a fixed, identical endowment y of babysitting time in each period. A family consumes only babysitting, and discounts the discrete stream of lifetime consumption, with each period's consumption weighted with a stochastic marginal utility: utility at time t is

$$E_t \sum_{s=0}^{\infty} \beta^s \theta_{t+s} c_{t+s}. \tag{3.1}$$

where β is the discount factor, c_t is consumption of babysitting at time t, and θ_t is a stochastic taste parameter. The linearity of consumption means that a nonnegativity constraint for consumption must be explicit:

$$c_{t+s} \geq 0, \qquad s \geq 0. \tag{3.2}$$

The taste shock, θ, is identically and independently distributed in each period as follows:

$$\theta \sim F(\cdot), \qquad 0 \leq \underline{\theta} \leq \theta \leq \overline{\theta} < \infty.$$

The realizations are assumed independent across time for each family. When a family has a low realization of θ, it will wish to store its time endowment in anticipation of realizations of high marginal utility. This desire to substitute consumption through time is the central force in the model. Because babysitting time is nonstorable, exchange of some sort is the only hope for such intertemporal substitution.

The realizations of θ are assumed independent across families as well as across time for each family, and the law of large numbers is assumed to hold. [3.1] This is the source of the potential for exchange in the

[2.2] In my own experience it was a bit more complicated: tickets were borrowed and loaned, and occasionally purchased with cash.

[3.1] That is, there is no aggregate uncertainty because the cross-sectional distribution of families' θ realizations replicates $F(\cdot)$ in every period. Judd (1985) shows that it is valid to assume this.

cooperative: parents who are "desperate" (ie, have a high θ realization, as when one child needs to go the the doctor and another must be left at home) will wish to maximize their consumption, while those who are not desperate should, in a well-behaved equilibrium, be willing to transfer consumption to the desperate in anticipation of a quid pro quo for their own future need.

My analysis will rise or fall on the existence of the value function, as I use a dynamic programming approach. In my narrow technical framework, utility functions are unbounded, and therefore the arguments that have been set out in more familiar dynamic programming models don't directly apply here. Nevertheless, the existence of the value function can be demonstrated, and indeed the solutions form a new class of closed form solutions that are of independent technical interest.

The existence issue arises because the optimization problem of grandparents and individuals is an extension of the following very simple one.

$$\max_{c_t, k_t} \sum_{s=0}^{\infty} \beta^s \theta c_{t+s} \tag{3.3}$$

subject to

$$c_t + k_{t+1} \leq \phi k_t + y_t \tag{3.4}$$

where β is a fractional discount factor, θ is a constant, c_{t+s} is consumption, k_t is current capital (or the stock of some asset like money or bonds), y_t is some endowment and ϕ is the gross marginal product of capital. There are nonnegativity constraints:

$$c_t \geq 0, \tag{3.5}$$

$$k_{t+1} \geq 0. \tag{3.6}$$

Finally, the initial stock of capital, k_t, is given.

This looks like a standard capital theory problem except for the linearity of utility and the production function. It has a trivial solution for the policy: if $\phi < \beta^{-1}$, consume all capital immediately and obtain zero income and utility thereafter; if $\phi > \beta^{-1}$, then always save. And if $\phi = \beta^{-1}$, optimal behavior is indeterminate. The value is clearly defined in the first case: it is $\theta k_t + \sum_{s=0}^{\infty} \beta^s \theta y_{t+s}$. When $\phi = \beta$, the value is infinite. When $\phi > \beta^{-1}$, no consumption ever occurs since it is always productive to save. The value of discounted utility is infinity in that case, since the stock of capital grows faster than it is discounted. Yet there is a paradox in that the individual with such a value would never actually consume any output and would hence have a value of zero. It is better to simply say that the value is undefined.

Things are not so cut and dried if θ can change over time. If the parameter ϕ is below β^{-1}, and θ takes on a low value $\underline{\theta}$ for T periods and a high value $\overline{\theta}$ thereafter, the optimal policy would be to save for T periods and then consume thereafter. The value function and policy would exist and be well defined in every period. Extending this logic, θ could fluctuate between $\underline{\theta}$ for T periods and $\overline{\theta}$ for U periods indefinitely. The existence of the value function and the optimal policy would then be more delicate. There must be some argument that T, U and ϕ interact so that the sum in (3.3) converges; this requires a restriction on the growth rate of consumption, and this is equivalent to requiring a bound on ϕ that is more complicated than β^{-1}.

In the sequel, θ fluctuates in this way, but stochastically, and moreover the analogue of ϕ is endogenous. I show that equilibria exist when conditions lead to satisfying the stochastic analogue of the growth condition for ϕ. Those conditions arise in monetary equilibria in which inflation is higher than is physically efficient. They arise in credit equilibria in which borrowing constraints are tighter than is physically efficient. And they arise in an optimal mechanism that redistributes endowment in a way that is physically inefficient. When these conditions cannot arise, I call that the nonexistence of equilibrium.

An important difference between this and standard bounded-utility dynamic programming is that undefined values and nonexistence of equilibrium are possible, and therefore some work has to be done to find the boundaries of these unpleasant possibilities. When the work is done, these boundaries have economic meaning.

4. A PRIVATE INSURANCE CONTRACT

If each family in the cooperative had a set of grandparents nearby, they could satisfy their babysitting needs through a long term contract with the grandparents. I will now develop the mechanics of a contract provided by grandparents to a single family in the above setting. I will later show that the set of such contracts includes as a subcase the efficient cooperative contract. I then use this as a benchmark against which to assess cooperative babysitting arrangements such as credit and ticket equilibria.

The grandparents are risk neutral and are in that sense like a "classical" principal; nor do they undergo taste shocks like the family. I speak of the grandparents as a a principal rather than an intermediary in order to emphasize that the contract is between the grandparents and a single family, without regard for the aggregate time resource constraint of the cooperative. Risk neutrality yields unchanging marginal valuations under statistical expectation; statistical expectation corresponds to averaging demands across families in determining prices.

The grandparents have access to infinite resources at a fixed discount factor, α, but this discount factor is not necessarily the same as β, that of the family. The grandparents can therefore sell insurance to the family by storing babysitting time when the family's realized need is low, and releasing it when it is high. I use the term "insurance" because of the family's desire to offset uncertainty, but the desired outcome of the insurance is not pure smoothing as in the case of concave utility, but the opposite: binging when need is high.

The grandparents' discounted return is:

$$E_t \sum_{s=0}^{\infty} \alpha^s (y - c_{t+s}),$$

that is, the return is the excess of the family's income over consumption kept by the grandparents, discounted at α. Imitating the recursive structure used by Green (1987, section 8) and by Thomas and Worrall, the dynamic programming recursion of this value is

$$W(V) = \sup_{\gamma(\cdot), V'(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \{y - \gamma(\theta, V) + \alpha W(V'(\theta, V))\} dF(\theta), \tag{4.1}$$

subject to the constraints

$$V = \int_{\underline{\theta}}^{\overline{\theta}} \{\theta\gamma(\sigma, V) + \beta V'(\sigma, V)\} dF(\theta) \tag{4.2}$$

$$\theta\gamma(\theta, V) + \beta V'(\theta, V) \geq \theta\gamma(\theta^*, V) + \beta V'(\theta^*, V), \qquad \theta^* \neq \theta \tag{4.3}$$

$$\gamma(\sigma, V) \geq 0 \tag{4.4}$$

$$V'(\sigma, V) \geq \underline{V} \tag{4.5}$$

$$V \quad \text{given.} \tag{4.6}$$

The grandparents indirectly solve the family's problem because the value function recursion of the family is a constraint, indeed it is the law of motion of the grandparents. The grandparents attempt to capture the family's time endowment by offering a combination of current consumption, $\gamma(\sigma, V)$, and a future value, $V'(\sigma, V)$, that yields some fixed value of expected utility, V. As the game evolves, this initial value of V is replaced by values V' that have been set in advance by the grandparents' contract, reflected in the recursion constraint (4.2). The grandparents would like to minimize the value they give the family, but are restrained

from doing so by the recursion constraint and the initial value. [4.1] These combinations of consumption and expected value respond to a signal, σ, the family sends about its realized state of need.

Extracting information about the family's state, which the grandparents cannot observe independently, is the only nontrivial task faced by the grandparents. Without some restriction, the family would always report desperate need if it generated a positive payoff. The information extraction task is accomplished by imposing incentive compatibility constraints, (4.3), requiring that the value of reporting falsely is less than the value of reporting truthfully.

If there were no uncertainty and $\alpha > \beta$, the grandparents would lend to the family; because of the linearity of the returns, the grandparents would initially lend an infinite amount to the family, collecting the high-interest return in the infinite future. With uncertainty, this degeneracy does not appear because of the necessity of the grandparents' extracting information from the family. Thus the excess of the grandparents' discount factor over that of the family can be thought of as an information extraction premium, corresponding in some sense to a risk premium.

The timing of actions implicit in the statement of the problem is as follows. First, the grandparents set the consumption and value functions, and the choices are known to the family. Then, the taste shock, θ, is realized. The family sends a signal, σ, which states the value of the shock truthfully: $\sigma = \theta$. The grandparents collect consumption from the family if a low θ is reported, and give consumption to the family if θ is high. The process then begins anew, with V replaced by $V'(\theta, V)$.

The next constraint, (4.4), is simply the non-negativity of consumption constraint that reflects the unavailability of storage for the family. The constraint (4.5), restricts the grandparents from offering extremely low future values in exchange for high current consumption. Hitting $\underline{V}$ corresponds to a family (a desperate one) spending all its money or borrowing to the hilt. $\underline{V}$ is not necessarily an absorbing state; a family that is not desperate can have its value state increased by the grandparents as a reward for refraining from consuming babysitting. While no motivation of the level of $\underline{V}$ is offered at this stage, I show in section 8 that when an additional restriction corresponding to aggregate feasibility is imposed, the value of $\underline{V}$ is determined as an endogenous function of the stochastic structure and taste parameters.

5. THE SOLUTION OF THE THRESHOLD VALUE OF θ

The grandparents' policy functions have a monotonicity property which greatly simplifies the sets over which family value states stochastically evolve. There are two sets of values of θ: low and high. Within each set there is a fixed strategy, and across the sets the incentive constraints hold in a particular way. The sets necessarily have a monotonic relationship with each other. The boundary dividing these sets, θ^*, along with the fixed policies that apply within the two sets, completely characterize the policy functions. The analysis can focus on finding these three numbers.

I show this property in Appendix A. Thus there is a cutoff value of θ, θ^*, such that below that point a single policy is followed by the grandparents and vice versa:

$$\gamma(\theta) = \begin{cases} 0, & \underline{\theta} \leq \theta < \theta^* \\ \overline{\gamma}, & \theta^* \leq \theta < \overline{\theta} \end{cases} \tag{5.1}$$

$$V'(\theta) = \begin{cases} \overline{V}, & \underline{\theta} \leq \theta < \theta^* \\ \underline{V}, & \theta^* \leq \theta < \overline{\theta} \end{cases} \tag{5.2}$$

with $\overline{\gamma} \geq 0$, and $\overline{V} \geq \underline{V}$. The intuitive reason the solutions have this form is as follows. First, it is efficient to provide babysitting in high-θ states. Some compensation must be taken for that provision in order to maintain incentive compatibility, and that compensation is a reduction of the future value of babysitting for the family. Conversely, the family's future value is increased if it has a low θ.

[4.1] The idea of using another agent's utility as a state variable in a recursion dates to Lucas and Stokey (1984); see also Stokey and Lucas (1989), pp 117 and 495. An additional reference is Abreu, Pierce and Stacchetti (1986). Green and Thomas and Worrall also made extensive use of the idea.

Within each of the sets $\underline{\theta} \leq \theta < \theta^*$ and $\theta^* \leq \theta < \overline{\theta}$, the reason that a nonconstant reward would fail to be incentive compatible is that the marginal incentive to lie is proportional to θ, which is nonconstant. The marginal incentive to lie is

$$\theta \frac{\partial}{\partial s}\gamma(s,V) + \beta \frac{\partial}{\partial s} V'(s,V)$$

and it must be zero in an incentive compatible contract. Since θ ranges over a continuum, the only way to do this is to make both $\frac{\partial}{\partial s}\gamma(s,V)$ and $\frac{\partial}{\partial s}V'(s,V)$ both zero except at discrete points. What remains is to find those discrete points, and then show that there is at most one of them. I show all this formally in the appendix.

With the structure of the policies determined, the next step is to find the actual values of the three policy parameters in terms of the fundamentals of the economy. In order to do that, it will be necessary to find the value function for the principal. The first step in this process is to note that the incentive constraints now boil down to two equations, one for each θ-set:

$$\beta\overline{V} \geq \theta\overline{\gamma} + \beta\underline{V} \qquad \underline{\theta} \leq \theta < \theta^*,$$

$$\theta\overline{\gamma} + \beta\underline{V} \geq \beta\overline{V} \qquad \theta^* \leq \theta < \overline{\theta}.$$

Both constraints will be satisfied if and only if the hunger threshold θ^* satisfies the equation

$$\theta^*\overline{\gamma} + \beta\underline{V} = \beta\overline{V}. \tag{5.3}$$

Using this information, the grandparents' maximum problem can now be restated in these terms:

$$\max_{\theta^*,\overline{\gamma},\underline{V}} \{y - (1 - F(\theta^*))\overline{\gamma} + \alpha F(\theta^*)W(\overline{V}) + \alpha(1 - F(\theta^*))W(\underline{V})\} \tag{5.4}$$

subject to

$$V = \int_{\theta^*}^{\overline{\theta}} \theta\overline{\gamma}dF(\theta) + \beta F(\theta^*)\overline{V} + \beta(1 - F(\theta^*))\underline{V} \tag{5.5}$$

and the binding incentive constraint

$$-\theta^*\overline{\gamma} + \beta(\overline{V} - \underline{V}) = 0. \tag{5.6}$$

For notational convenience, I now define the quantities $\Phi(\theta^*) \equiv \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta)$ and $\mathcal{E}(\theta^*) \equiv F(\theta^*)\theta^* + \Phi(\theta^*)$. (The notation $\mathcal{E}$ is purposely meant to evoke statistical expectation: if F is Bernoulli, and θ^* is between the two realizations, then $\mathcal{E} = E(\theta)$.) In matrix form the two constraints then take the form

$$\begin{pmatrix} \overline{\gamma} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} \Phi(\theta^*) & \beta F(\theta^*) \\ -\theta^* & \beta \end{pmatrix}^{-1} \begin{pmatrix} V - \beta(1 - F(\theta^*))\underline{V} \\ \beta\underline{V} \end{pmatrix}. \tag{5.7}$$

with solution

$$\overline{V} = \left(1 - \beta\frac{\theta^*}{\beta\mathcal{E}(\theta^*)}\right)\underline{V} + \frac{\theta^*}{\beta\mathcal{E}(\theta^*)}V, \tag{5.8}$$

$$\overline{\gamma} = \frac{1}{\mathcal{E}(\theta^*)}(V - \beta\underline{V}). \tag{5.9}$$

The problem of finding the optimal threshold value of θ^* can be considered separately from that of finding the optimal functions γ and V'. This requires first finding the value function. The technique will be to make a good initial guess and then verify that it has appropriate properties. In this endeavor, posit that W is affine in the current value state: $W(V) = W_0 + W_V V$. In Appendices B and C I show that this function solves the dynamic programming recursion and that it is the grandparents' value function. In Appendix B I show that the grandparents' Bellman equation recursion devolves into an Euler equation in θ^*,

$$\theta^* = \frac{\beta}{\alpha}\mathcal{E}(\theta^*). \tag{5.10}$$

This equation, and therefore θ^*, is independent of the current value state, V, ensuring that the solution is constant over time.

PROPOSITION 5.1 *If F is nonatomic and $\alpha > \beta$, then there is a unique $\theta^* \in [\underline{\theta}, \overline{\theta}]$.*

The proof, which is in Appendix B, makes use of the properties of $\mathcal{E}$ in a simple fixed point argument. With the solution in hand, I now turn to characterizing the stochastic processes of consumption and the family's value state. Using the intuition that W is decreasing in V, the solutions of the maximum problem can be stated easily: consumption will be zero if the family signals nondesperation, and will be at a maximum if desperation is signalled. Thus in the case of desperation, the family's value state will revert to its minimum.

In the case of nondesperation, the family's value state grows according to the difference equation (5.8), with the rate of growth of the value determined by the coefficient of V, $\theta^*/\beta\mathcal{E}(\theta^*)$. Since consumption is an affine function of V according to (5.8), it is necessary to show that the rate of growth is bounded sufficiently for the value to be well defined, as discussed in section 3. I turn to this analysis in the next section.

6. THE DISTRIBUTION OF STATES AND THE GRANDPARENTS' VALUE

I will now show when the optimum for the grandparents exists by characterizing the stochastic process for each family's state and associated babysitting consumption, and then establish when the sum characterizing discounted expected profit for the grandparents converges. If the grandparents are too patient, the grandparents will provide unlimited babysitting, yet never collect anything in return. The sum characterizing discounted profit does not converge, and is then undefined in the sense given in the introduction. Even if it were assumed that the grandparents provided babysitting in this circumstance, it would not characterize a contract that required no outside resources.

Because the family reverts to $\underline{V}$ with probability $(1 - F(\theta^*))$, the set of states value states is discrete and is described by restating (5.8) as a difference equation: [6.1]

$$V_k = (1 - \beta\rho)\,\underline{V} + \rho V_{k-1}, \tag{6.1}$$

$$V_0 = \underline{V},$$

where I define the growth coefficient, ρ, by

$$\rho \equiv \theta^*/\beta\mathcal{E}(\theta^*). \tag{6.2}$$

Observe that $0 \le \rho \le \beta^{-1}$ and is a nondecreasing function of θ^*.

The need states will have a distribution, Ψ, with probabilities ψ_k. Associated with each state is a consumption level, γ_k. From equation (5.9),

$$\gamma_k = -\frac{\beta}{\mathcal{E}(\theta^*)}\underline{V} + \frac{1}{\mathcal{E}(\theta^*)}V_{k-1}, \qquad k = 1, \ldots. \tag{6.3}$$

The probabilities ψ_k of state V_k are

$$\psi_k = (1 - F(\theta^*))F(\theta^*)^k, \qquad k = 0, \ldots. \tag{6.4}$$

That is, a family is desperate and reverts to state zero with probability $(1 - F(\theta^*))$ regardless of its present state, obtaining value $\underline{V}$. Subsequent nondesperation occurs regardless of state with probability $F(\theta^*)$ with the value state V_k rising to V_{k+1}. thus there is a long run probability $F(\theta^*)(1 - F(\theta^*))$ of the family being in state 1, $F(\theta^*)^2(1 - F(\theta^*))$ for state 2, and so on.

[6.1] Not quite: the family's initial value could start out of the set of discrete values generated by the solution for the grandparents' policy. However it is straightforward to show that the distribution of states Ψ converges strongly; I do this in the appendix.

The solution of the difference equation for V_k is

$$V_k = (1-\rho^k)V^s + \rho^k \underline{V}, \tag{6.5}$$

where V^s is the stationary solution,

$$V^s \equiv \frac{1-\beta\rho}{1-\rho}\underline{V}. \tag{6.6}$$

Substituting from (6.5) into (6.3),

$$\gamma_k = \frac{1}{\mathcal{E}(\theta^*)}\left(-\beta\underline{V} + ((1-\rho^{k-1})V^s + \rho^{k-1}\underline{V})\right).$$

The probability of γ_k is the probability of state V_{k-1} times the probability of desperation:

$$\Pr(\gamma_k) = \psi_{k-1}(1-F(\theta^*)).$$

The grandparents' expected utility must be convergent. If the grandparents start a family with value $\underline{V}$, then the expected payout by the grandparents is the sum of the probabilities of payout γ_k, discounted by α^k, the discount factor relevant for the realization. Expected profit is

$$\sum_{k=1}^{\infty}\alpha^k\psi_k(y-\gamma_k) = \sum_{k=1}^{\infty}(1-F(\theta^*))(\alpha F(\theta^*))^k(y-\gamma_k). \tag{6.7}$$

The conditions needed for convergence are

$$\alpha F(\theta^*) < 1, \qquad \alpha F(\theta^*)\rho < 1.$$

The following proposition shows that these conditions are generically satisfied.

PROPOSITION 6.1 *If F is nonatomic, $1 > \alpha > \beta$ are sufficient conditions for the fixed point of the dynamic programming problem, W, to be the value of the grandparents' problem.*

PROOF: From the first order condition for θ^*, (5.10), the optimal value of θ^* is such that $\rho\alpha = 1$, and since $\alpha < 1$, convergence of (6.7) is guaranteed if $F(\theta^*) < 1$. By proposition 5.1, $\theta^* < \overline{\theta}$ and hence $F(\theta^*) < 1$. The affine solution to the grandparents' dynamic programming recursion is therefore defined. The hypothesis $\alpha > \beta$ satisfies the growth condition in lemma C.1, and by proposition C.2, the solution of the dynamic programming problem is therefore the value function. ∎

The intuition of the proof is that the conditions of the proposition provide a bound on the growth rate of the family's value state, and this growth rate is dominated by the discounting of the grandparents.

7. THE ZERO-PROFIT FRONTIER

A social insurance contract would begin with families in a common state, maximizing the expected utility of a representative family, subject to revelation and to feasibility. The grandparents so far have induced revelation. I now impose feasibility as well, the equivalent of requiring zero profit in Rothschild and Stiglitz's (1976) analysis of static insurance markets. Each value of α, the discount factor of the grandparents, is associated with a zero-profit equilibrium, and endogenously fixes a minimum value state, $\underline{V}$, a parameter that has up until now been exogenous.

The reason for focusing attention on zero profit equilibria is that they have the potential to be equivalent to optimal mechanisms. In a partial equilibrium setting this would be immediate, but a more formal demonstration is needed here.

One could contemplate optimal contracts directly, as Atkeson and Lucas (1990) have done. There are two reasons for thinking about them indirectly as I am doing here. First, the indirect approach admits the possibility of inefficient equilibria. As a technical proposition this set is relatively easy to characterize.

One can look at the boundary of this set; the boundary will be the set of efficient contracts. Second, the discounting of the grandparents turns out to have a direct and appealing connection to interest rates in equilibrium. One can map the search for an equilibrium interest rate into the search for a discount factor. Since grandparent contracts exist for many discount factors, the search for the "correct" discount factor that corresponds to an equilibriun can be deferred to a later stage of analysis.

A feasible program uses no outside resources:

$$E(\gamma) \leq y.$$

Since there is no satiation here, it is relevant to focus on contracts in which there is no waste of resources so that equality holds. Calculating the expectation, the condition is

$$\begin{aligned} y &= \sum_{k=1}^{\infty}(1 - F(\theta^*))^2 F(\theta^*)^{k-1}\frac{1}{\mathcal{E}(\theta^*)}\{-\beta\underline{V} + ((1-\rho^{k-1})V^s + \rho^{k-1}\underline{V})\} \\ &= \frac{(1-F(\theta^*))^2}{\mathcal{E}(\theta^*)}\left\{\frac{V^s - \beta\underline{V}}{1-F(\theta^*)} + \frac{\underline{V} - V^s}{1-\rho F(\theta^*)}\right\}. \end{aligned} \tag{7.1}$$

If the sum converges, the equation can be solved for the minimum value state, $\underline{V}$:

$$\underline{V} = \frac{1-\rho F(\theta^*)}{1-F(\theta^*)}\mathcal{E}(\theta^*)\frac{y}{(1-\beta)}. \tag{7.2}$$

Welfare of the family. A Rawlsian welfare notion starts a family in a cooperative with random initial expected utility, with the probability distribution of that initial expected utility identical to its long-run stationary probability distribution. Contractual welfare is the average of this initial state over the stationary distribution, that is,

$$\sum_{k=0}^{\infty}\psi_k V_k = \sum_{k=0}^{\infty}(1-F(\theta^*))F(\theta^*)^k\{(1-\rho^k)V^s + \rho^k\underline{V}\} \tag{7.3}$$

To calculate this value, substitute for $\underline{V}$ from the zero-profit condition (7.2), yielding

$$E(V) = \frac{\Phi(\theta^*)}{1-F(\theta^*)}\frac{y}{1-\beta} \tag{7.4}$$

if the sum converges. The upper bound of this expected value is $\bar{\theta}y/(1-\beta)$. At this upper bound families receive utility equal to that which they could receive by consuming all their endowment as if in their most desperate state. [7.1] This is preferable to autarky, which would yield an expected value of only $E(\theta)y/(1-\beta)$, since the conditional mean $\Phi(\theta^*)/(1-F(\theta^*))$ exceeds $E(\theta)$. However the maximum of this utility depends on θ^*, which in turn influences whether the sum is convergent. In fact, welfare is clearly increasing in θ^*. But the Rawlsian ideal welfare cannot be attained:

PROPOSITION 7.1 *If F is nonatomic, $\theta^* < \bar{\theta}$ and the family's welfare in the zero-profit contract is bounded strictly below $\bar{\theta}y/(1-\beta)$.*

PROOF: The condition $\rho F(\theta^*) < 1$ is necessary for the convergence of (7.1). Combined with the first order condition $\rho\alpha = 1$ from (5.10) and (6.2), this puts an upper bound on $F(\theta^*)$:

$$\alpha = 1/\rho \geq F(\theta^*). \tag{7.5}$$

Expected family welfare is maximized when the two conditions $\alpha = \beta$ and $F(\theta^*) = 1$ both hold; the last inequality says this is impossible.

[7.1] For a nonatomic probability distribution F, the upper bound of utility could not be implemented—it involves each familiy having an infinite waiting time for the most desperate state.

The condition needed for convergence, namely $\rho F(\theta^*) < 1$, means maximum welfare is attained when $F(\theta^*) = 1/\rho = \alpha$. This results in the equation for the maximum feasible θ^*,

$$(1-\beta)\theta^* F(\theta^*) = \beta\Phi(\theta^*). \tag{7.6}$$

The left hand side of this equation is monotone increasing in θ^*, ranging from zero to $(1-\beta)\overline{\theta}$. The left hand side is monotone decreasing (because $\Phi'(\theta) = -\theta F'(\theta)$) ranging from $\beta E(\theta)$ to zero. Therefore a unique interior solution exists. ∎

The solution and actual welfare will depend on the particular distribution used. Examples were presented in my 1988 paper.

The growth factor ρ is the implicit rate of return available to the grandparents for investing in the value state of the family. This rate of return is an increasing function of θ^*—as the range of states which provide no babysitting grows, the rate of return needed to compensate families for foregoing sitting increases. The grandparents pay a high rate of return to most families who are savers. To maintain zero profit, they must extract payments from families that cash in their babysitting values by setting the minimum value, $\underline{V}$, very low. Beyond a threshold this value becomes negative. Because of the recursive definition of the family's value, the minimum value can be negative only if the family endures negative consumption, which is infeasible.

These negative findings might be mitigated if the grandparents' contract could be bettered by a cooperative mechanism. The following theorem argues that this cannot be done.

THEOREM 7.2 *No feasible, incentive-compatible cooperative mechanism provides greater welfare than the maximal zero profit contract.*

PROOF: Denote the average welfare level in the zero-profit equilibrium $V_0(y)$. Suppose such a mechanism existed. It would supply an average welfare level, $V_M(y)$, greater than $V_0(y)$. The mechanism is independent of the endowment level; in particular it could be imposed on an economy identical except for having a lower income and consequent lower welfare. Choose a level of income $y' < y$ such that $V_M(y') = V_0(y)$. Now let the grandparents operate the mechanism, offering the same level of welfare as before, $V_0(y)$, after extracting $y' - y$ from the family in each period. Then they now make positive profit, which contradicts the assumption that the grandparents had maximized profit. The grandparents could attain zero profit by returning $y' - y$ to the family, however this would contradict the maximality assumption. ∎

The zero profit frontier is therefore the efficiency frontier for the cooperative's mechanism.

8. CREDIT

In a large babysitting cooperative without grandparents, there is nevertheless potential for exchange. Families that are desperate will wish to maximize their consumption of babysitting, while those that are not should, in a well-behaved equilibrium, be willing to lend to the desperate in anticipation of their own future need. I now construct a credit equilibrium in this setting in preparation for addressing this point.

The mechanics of the credit equilibrium are as follows. Each family is characterized by a state of credit holding, B. That state fluctuates stochastically, falling when babysitting need is high and rising when it is low. Not surprisingly, families follow a rule that they save when their babysitting need is below a threshold, and borrow when it is above the threshold. Because utility is linear, when babysitting needs are high, the family "blows its wad" and then starts over with the most debt they are permitted. With each family undergoing different shocks, there is a distribution of debt states that must be zero in the aggregate. The interest rate is set to ensure that this condition is met.

In a credit equilibrium, families can borrow and lend at a fixed interest rate. Expressing the family's problem in recursive form,

$$V(B,\theta) = \max_{c,B'}\{\theta c + \beta EV(B',\theta')\} \tag{8.1}$$

subject to

$$c + B' \leq y + rB, \tag{8.2}$$

$$c \geq 0. \tag{8.3}$$

where r is the *gross* interest rate, and B' is bonds held into the next period; borrowers have negative bond holdings.

In addition, there is a borrowing constraint that limits debt to a function of wealth. It is only necessary to limit borrowing, since saving is limited by available endowment:

$$B' \geq \frac{1 - r^{n^*+1}}{1 - r} y, \quad n^* > -\infty, \tag{8.4}$$

with n^* potentially negative. As n^* tends toward minus infinity, the borrowing constraint tends toward discounted wealth in the usual way. The starting point for the constraint is to note that a family that started with no bonds and that was repeatedly nondesperate would accumulate debt geometrically:

$$\{B_n\}_{n=0}^{\infty} = \{y, ry + y, r(ry + y) + y, \cdots\},$$

and

$$B_n = \sum_{j=0}^{n} r^j y = \frac{1 - r^{n+1}}{1 - r} y, \quad n \geq n^*, \tag{8.5}$$

where n indexes the number of periods elapsed in a continually nondesperate state, with $n = 0$ arbitrarily chosen to index the initial period of low urgency. Similarly, a family that started out in debt and was repeatedly nondesperate would spend all income on repayment and repay the debt in increments of size $r^n y$, with n negative. The fundamental restriction expressed in (8.4) with $n^* > -\infty$ is that the family must be able to repay its debt in a finite time if it is continually nondesperate. If $n^* = -\infty$ were permitted, borrowers could never repay their debt.

Solution of the family's problem. As when solving the problem of the grandparents, it is extremely helpful to first establish that the solution of the Bellman equation has a particular convenient form and then establish that it also is the value function. As with the grandparents' problem, it is easy to see that the solution to the Bellman equation is affine in bonds. I defer to appendix D the proof that it is the value function.

$$V(B, \theta) = A_0(\theta) + A_B(\theta)B. \tag{8.6}$$

Linear progamming methods can now be used to find the optimum. The solution for consumption is

$$g(\theta, B) = c = \begin{cases} 0 & \theta \leq \beta \overline{A}_B \\ y + rB - \frac{1 - r^{n^*+1}}{1-r} y, & \theta > \beta \overline{A}_B \end{cases} \tag{8.7}$$

where $\overline{A}_B = EA_B$. Thus when a family is desperate, it can consume its endowment plus interest (or less interest if it is a debtor) and in addition nearly the present value of all future endowments. The solution for B' is

$$h(\theta, B) = B' = \begin{cases} rB + y & \theta \leq \beta \overline{A}_B \\ \frac{1 - r^{n^*+1}}{1-r} y & \theta > \beta \overline{A}_B \end{cases} \tag{8.8}$$

Substituting the solutions into the functional equation defined by (8.6) yields the pair of equations

$$A_0(\theta) + A_B(\theta)B = \begin{cases} \beta \overline{A}_0 + \beta \overline{A}_B(rB + y) & \theta \leq \beta \overline{A}_B \\ \theta(y + rB - \dfrac{1 - r^{n^*+1}}{1 - r} y) & \\ \quad + \beta \overline{A}_0 + \beta \overline{A}_B \dfrac{1 - r^{n^*+1}}{1 - r} y & \theta > \beta \overline{A}_B \end{cases} \tag{8.9}$$

Equating coefficients of B^0 and B^1 and simplifying yields the pair of equations

$$A_0(\theta) = \begin{cases} \beta \overline{A}_0 + \beta \overline{A}_B y & \theta \leq \beta \overline{A}_B \\ \theta y + \beta \overline{A}_0 - \dfrac{1 - r^{n^*+1}}{1 - r} y(\theta - \beta \overline{A}_B) & \theta > \beta \overline{A}_B \end{cases}$$

$$A_B(\theta) = \begin{cases} \beta\overline{A}_B r & \theta \leq \beta\overline{A}_B \\ \theta r & \theta > \beta\overline{A}_B \end{cases}$$

The marginal value of bonds, $A_B(\theta)$, is defined solely in terms of its own future average. Averaging and simplifying yields an equation in this average:

$$\overline{A}_0 = \frac{y}{1-\beta}\Bigg\{\beta\overline{A}_B F(\beta\overline{A}_B) + \int_{\beta\overline{A}_B}^{\overline{\theta}} \theta dF(\theta) - \frac{1-r^{n^*+1}}{1-r}\left(\int_{\beta\overline{A}_B}^{\overline{\theta}} \theta dF(\theta) - \beta\overline{A}_B(1-F(\beta\overline{A}_B))\right)\Bigg\} \tag{8.10}$$

$$\overline{A}_B = r\{\beta\overline{A}_B F(\beta\overline{A}_B) + \int_{\beta\overline{A}_B}^{\overline{\theta}} \theta dF(\theta)\} \tag{8.11}$$

PROPOSITION 8.1 *If $\beta r < 1$, there is a unique solution to (8.10-11).*

PROOF: Multiply (8.11) by β; it then duplicates the grandparents' Euler condition for θ^*, and proposition 5.1 applies. Equation (8.10) can then be written more succintly as

$$\overline{A}_0 = \frac{y}{1-\beta}\beta\overline{A}_B\left\{\frac{1}{\beta r} - \frac{1-r^{n^*+1}}{1-r}\left(\frac{1}{\beta r}-1\right)\right\} \tag{8.10'}$$

Substitute the solution of (8.11) in the right hand side of this equation. ∎

I now set out a standard definition of a credit equilibrium. I focus only on stationary equilibrium here, but in the appendix I argue that nonstationary equilibria don't change the analysis. There are two kinds of nonstationary equilibria to contemplate. In the first, some initial distribution changes over time but eventually converges to the stationary distribution I characterize here. In the second, some initial distribution changes but does not converge. Both types are possible here. In Appendix F I show that the convergent type is generic and converges strongly to the stationary equilibrium. In Appendix G I show that the nonstationary equilibrium exists at the boundary of the set of all equilibria, but that its nonstationarity does not affect the implicit rate of return. [8.1]

DEFINITION 8.1 *A credit equilibrium is a sequence of consumption policy functions $g_t(\cdot,\cdot) : R_+ \times [\underline{\theta}, \overline{\theta}] \to R$, credit holding policy functions $h_t(\cdot,\cdot) : [\underline{B}, \infty) \times [\underline{\theta}, \overline{\theta}] \to R$, gross rates of return on credit r_t, borrowing constraints $\underline{B}_t$, and distribution functions Ψ_t, such that*

(i) the functions g_t and h_t solve the individual's optimization problem (8.1-4);

(ii) the goods market clears so that average consumption equals average endowment:

$$\int g_t(B,\theta)d\Psi_t(B)dF(\theta) = y; \tag{8.12}$$

(iii) the credit market clears so that average credit held equals zero:

$$\int h_t(B,\theta)d\Psi_t(B)dF(\theta) = 0 \tag{8.13}$$

(iv) the distribution of credit balances has the law of motion determined by the transition rules for credit holding:

$$\Psi_{t+1}(B') = \int\int_{\{(B,\theta):B\geq\underline{B},h(B,\theta)\leq B'\}} d\Psi_t(B)dF(\theta) \tag{8.14}$$

[8.1] Atkeson and Lucas develop machinery that handles this nonstationarity. It is important to handle it because it emerges generically in efficient equilibria.

PROPOSITION 8.2 *Let $\beta r < 1$. Then there is an $n^* > -\infty$ such that a stationary credit equilibrium exists.*

PROOF: The optimality of the policy functions g and h was established in proposition 8.1, satisfying condition (i) of definition 8.1.

Because of the discrete structure of the support of Ψ, it is possible to rewrite the transition rule (8.14) as

$$\Psi(B') = F(\beta \overline{A}_B)\Psi(\frac{B'-y}{r}) + (1 - F(\beta \overline{A}_B)). \tag{8.15}$$

The first term corresponds to families that transit into state B' or lower from nondesperation by buying an additional unit of bonds or paying off part of their debt. The second term corresponds to families that are desperate and borrow up to the borrowing constraint. The equation is similar to Lucas's (1980) equation (11) and to equation (17) in Taub (1988) in that families can "spend down" their bond accumulation all at once. Recalling that bondholdings are indexed by sequences of consecutive nondesperate periods in (8.5), (8.15) can be written as an ordinary first-order difference equation:

$$\Psi_n = F(\beta \overline{A}_B)\Psi_{n-1} + (1 - F(\beta \overline{A}_B)), \quad n \geq n^*,$$

with solution

$$\Psi_n = 1 - F(\beta \overline{A}_B)^{n-n^*+1}. \tag{8.16}$$

This solution is stable in the sense that if the economy "starts up" with an arbitrary distribution of debt satisfying (8.12), it will converge strongly to this equilibrium debt distribution; a proof of this is given in appendix F. This establishes (iv).

Thus in the long run there are many debtors close to their borrowing limit and creditors with widely dispersed wealth levels. With the distribution of debt explicit, it is possible to explore the relation between the equilibrium interest rate and the borrowing restriction. Using (8.16), the probability of the discrete credit state B_n is:

$$\psi_n = \Psi_n - \Psi_{n-1} = F(\beta \overline{A}_B)^{n-n^*}(1 - F(\beta \overline{A}_B)) \tag{8.17}$$

which is identical to the probability in (6.3). Substituting in the zero aggregate debt equilibrium condition (8.13) from (8.17) and (8.5) yields the equilibrium condition

$$0 = \sum_{n=n^*}^{\infty} F(\beta \overline{A}_B)^{n-n^*}(1 - F(\beta \overline{A}_B))\frac{1-r^{n+1}}{r-1}y \tag{8.18}$$

If the sums converge, this reduces to

$$r^{n^*+1} = \frac{1 - rF(\beta \overline{A}_B)}{1 - F(\beta \overline{A}_B)}. \tag{8.19}$$

When combined with (8.11), this condition determines the equilibrium n^* for a given interest rate, satisfying the zero net debt condition (iii). The goods market clearing condition (ii) follows similarly by substitution of the consumption policy function g into the budget constraint. ∎

COROLLARY 8.3 *The equilibrium interest rate r is decreasing in n^*.*

The proof is in appendix E.

The proposition shows that relaxing the borrowing constraint is associated with increasing interest rates. It also shows that the equilibrium interest rate must be below the inverse of the subjective discount factor. This finding agrees with other work on heterogeneous-agent asset equilibria; recent examples include Clarida (1990) and Huggett (1991).

9. THE ZERO-PROFIT CONTRACT IS EQUIVALENT TO A CREDIT EQUILIBRIUM

I now demonstrate that solving the grandparents' problem is equivalent to solving the family's problem in a credit equilibrium, using duality analysis in the spirit of Green's analysis. This will enable me to show that

a credit equilibrium mimics a zero profit contract. To do this, I focus separately on the linear programming part of the problem and show that the dual problem of the grandparents can be manipulated to become the dual of the family's problem. I then show that the threshold value θ^* is the same for the grandparents' problem and the family's problem when the rate of return is equal to the discount rate of the grandparents.

For this purpose it is convenient to treat the grandparents' problem as one with just two Bernoulli distributed θ-shocks, θ_1 and θ_2. The correspondence with the continuum of shocks is as follows: $\theta_1 \equiv \theta^*$, $\theta_2 \equiv (1-F(\theta^*))^{-1}\Phi(\theta^*)$, $\Pr(\theta_1) \equiv \phi \equiv F(\theta^*)$. Also define $v = V - \underline{V}$ and $v_i = V_i - \underline{V}$, $i = 1, 2$; thus when V is at its minimum, v is zero.

Reversing the signs in the objective of the original grandparents problem and gathering all the constants to the left hand side of the recursion, it is possible to write it as a cost minimization problem: [9.1]

$$\min_{\gamma_i, V_i}\{\phi\gamma_1 + (1-\phi)\gamma_2 - \alpha W_V(\phi v_1 + (1-\phi)v_2)\}$$

subject to the value recursion constraint

$$\phi\theta_1\gamma_1 + (1-\phi)\theta_2\gamma_2 + \beta\phi v_1 + \beta(1-\phi)v_2 \geq v + (1-\beta)\underline{V},$$

and the incentive and nonnegativity constraints

$$\theta_1\gamma_1 - \theta_1\gamma_2 + \beta v_1 - \beta v_2 \geq 0,$$

$$\theta_2\gamma_2 - \theta_2\gamma_1 + \beta v_2 - \beta v_1 \geq 0,$$

$$\gamma_i \geq 0, \qquad v_i \geq 0,$$

The problem may be written in matrix form:

$$\min_{\gamma_i, v_i} \begin{pmatrix} \phi & (1-\phi) & -\alpha\phi W_V & -\alpha(1-\phi)W_V \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ v_1 \\ v_2 \end{pmatrix} \tag{9.1}$$

subject to

$$\begin{pmatrix} \phi\theta_1 & (1-\phi)\theta_2 & \beta\phi & \beta(1-\phi) \\ \theta_1 & -\theta_1 & \beta & -\beta \\ -\theta_2 & \theta_2 & -\beta & \beta \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ v_1 \\ v_2 \end{pmatrix} \geq \begin{pmatrix} v + (1-\beta)\underline{V} \\ 0 \\ 0 \end{pmatrix} \tag{9.2}$$

and the nonnegativity constraints.

Defining $x = (x_1, x_2, x_3)$, the dual problem is

$$\max_x \quad x^T \begin{pmatrix} v + (1-\beta)\underline{V} \\ 0 \\ 0 \end{pmatrix} \tag{9.3}$$

subject to

$$\begin{pmatrix} \phi\theta_1 & \theta_1 & -\theta_2 \\ (1-\phi)\theta_2 & -\theta_1 & \theta_2 \\ \beta\phi & \beta & -\beta \\ \beta(1-\phi) & -\beta & \beta \end{pmatrix} x \leq \begin{pmatrix} \phi \\ (1-\phi) \\ -\alpha\phi W_V \\ -\alpha(1-\phi)W_V \end{pmatrix} \tag{9.4}$$

$$x \geq 0. \tag{9.5}$$

[9.1] Readers may find the argument easier to follow by temporarily disregarding the incentive constraints.

Now define $z_1 = x_1^{-1}$, $z_2 = x_1^{-1}x_2$, and $z_3 = x_1^{-1}x_3$. Recalling that $v \equiv V - \underline{V}$, so $W(V) = W_0 + W_V(\underline{V} + v)$, it is natural to define $w = -W_V v$, and $\overline{W} = W_0 + W_V\underline{V}$. Thus $w = \overline{W} - W > 0$. The value of the left hand side of the cost minimization problem is then $y + w - (1-\alpha)\overline{W}$.

The problem of maximizing $x_1(v + (1-\beta)\underline{V})$ is the same as minimizing $z_1(y + w - (1-\alpha)\overline{W})$, since both $v + (1-\beta)\underline{V}$ and $y + w - (1-\alpha)\overline{W}$ are constants. Dividing the last two constraints by $-W_V$ and rearranging, the dual problem can therefore be written

$$\min_z \quad z^T \begin{pmatrix} y + w - (1-\alpha)\overline{W} \\ 0 \\ 0 \end{pmatrix} \tag{9.6}$$

subject to the reformulated constraint

$$\begin{pmatrix} \phi & -\theta_1 & \theta_2 \\ (1-\phi) & \theta_1 & -\theta_2 \\ \alpha\phi & \beta W_V^{-1} & -\beta W_V^{-1} \\ \alpha(1-\phi) & -\beta W_V^{-1} & \beta W_V^{-1} \end{pmatrix} z \geq \begin{pmatrix} \phi\theta_1 \\ (1-\phi)\theta_2 \\ -\beta\phi W_V^{-1} \\ -\beta(1-\phi)W_V^{-1} \end{pmatrix} \tag{9.7}$$

$$z \geq 0. \tag{9.8}$$

Call this the inverted dual problem. This problem is itself the dual of another problem: that of the family. The dual of the inverted dual is

$$\max_{\gamma_i, w_i} \begin{pmatrix} \phi\theta_1 & (1-\phi)\theta_2 & -\beta\phi W_V^{-1} & -\beta(1-\phi)W_V^{-1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ w_1 \\ w_2 \end{pmatrix} \tag{9.9}$$

subject to

$$\begin{pmatrix} \phi & (1-\phi) & \alpha\phi & \alpha(1-\phi) \\ -\theta_1 & \theta_1 & \beta W_V^{-1} & -\beta W_V^{-1} \\ \theta_2 & -\theta_2 & -\beta W_V^{-1} & \beta W_V^{-1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ w_1 \\ w_2 \end{pmatrix} \leq \begin{pmatrix} y + w - (1-\alpha)\overline{W} \\ 0 \\ 0 \end{pmatrix} \tag{9.10}$$

and nonnegativity constraints. (The "extra" constraints appear as a residual of the incentive constraints. They state the requirement that in any "interesting" credit equilibrium, the borrowing constraint must be binding in at least some states.)

LEMMA 9.1 *Let $\beta\alpha^{-1} < 1$. Then the solution to the family's problem (9.9-10) is as follows:*

$$\gamma_1 = 0, \quad w_1 = \alpha^{-1}(y + w - (1-\alpha)\overline{W});$$

$$\gamma_2 = y + w - (1-\alpha)\overline{W} \quad w_2 = 0.$$

PROOF: The binding incentive compatibility constraint in (9.10) states that families in a low state of need will not misrepresent themselves as desperate. The binding incentive constraint and the binding recursion constraint form the pair of equations

$$(1-\phi)\gamma_2 + \alpha\phi w_1 = y + w - (1-\alpha)\overline{W}$$

$$\theta_1\gamma_2 + \beta W_V^{-1} w_1 = 0$$

Recall that $\theta_1 = \theta^*$ by definition. Since $\beta\alpha^{-1} < 1$, equation (B.10) holds by proposition B.1, and therefore by corollary B.2, $W_V^{-1} = -\frac{\alpha}{\beta}\theta_1$. Substituting this formula into the two equations above yields the result. ■

PROPOSITION 9.2 *The value of problem (9.1-2) is $y + w - (1-\alpha)\overline{W}$ if and only if the value of problem (9.9-10) is $v + (1-\beta)\underline{V}$.*

PROOF: If the grandparents' problem is solved with a binding recursion constraint, then by duality the maximum expected value of the family's problem is $v + (1-\beta)\underline{V}$. If the family's problem is solved with a binding budget constraint then by duality the minimum expected cost of the grandparents' problem is $y + w - (1-\alpha)\overline{W}$. ∎

Now define credit as $B_i' = \alpha(w_i - \overline{W})$, and define the borrowing constraint by the case when the nonnegativity constraint for w_i is binding: $B_i \geq \underline{B} \equiv -\alpha\overline{W}$. Substituting the definition of the value B_i in the maximand, the family's problem becomes the credit problem

$$V = \max\{\phi\theta_1\gamma_1 + (1-\phi)\theta_2\gamma_2$$
$$+\beta\left(\phi(\underline{V} - W_V^{-1}(\overline{W} + \alpha^{-1}B_1')) + (1-\phi)(\underline{V} - W_V^{-1}(\overline{W} + \alpha^{-1}B_2')\right)\}$$

subject to

$$\phi\gamma_1 + (1-\phi)\gamma_2 + \phi B_1' + (1-\phi)B_2' \leq y + \alpha^{-1}B$$
$$B_i' \geq \underline{B}, \qquad \gamma_i \geq 0,$$

where the extra constant in the maximand, $-W_V^{-1}\overline{W}$, arises from the borrowing constraint.

These results make it possible to focus on the linear-programming duality separately from finding the optimal threshold of need in comparing the grandparents' contract and the family's optimum problem:

LEMMA 9.3 *The threshold value of θ is identical for the credit problem of a family and the grandparents' problem for $\alpha = r^{-1}$.*

PROOF: By proposition 9.1 and lemma E.1, the solution of the grandparents' problem is dual to that of the family if the threshold value of need, θ^*, is the same. Define $\theta^* \equiv \beta\overline{A}_B$. Then equation (8.11) for the value function of the family in a credit equilibrium is identical to the equation for the grandparents' optimal threshold, (5.10). ∎

PROPOSITION 9.4 *A zero-profit grandparents' contract is equivalent to a credit equilibrium such that the following conditions hold: (i) $\alpha = r^{-1}$; (ii) the stochastic process of consumption in the contract is identical to the stochastic process of consumption in the credit equilibrium; (iii) the distribution Ψ of value states in the contract is identical to the distribution of value states induced by the distribution of credit states Ψ in the credit equilibrium; (iv) the minimum expected value state $\underline{V}$ of the contract is identical to the minimum value state attainable in the credit equilibrium as determined by the borrowing constraint $\underline{B}$.*

PROOF: By lemma 9.3, the threshold value of θ is identical in the contract and the credit equilibrium. Therefore the consumption policy functions for the contract (5.9) and the credit equilibrium (8.7-8) are identical.

Solve the zero-profit condition (7.2) to yield

$$\frac{(1-\beta)\underline{V}}{\mathcal{E}(\theta^*)y} = \frac{1-\rho F(\theta^*)}{1-F(\theta^*)} \tag{9.11}$$

The right hand side is identical to the right hand side of the credit equilibrium condition (8.14). Recall the definition of ρ, that $\alpha = \rho^{-1}$ in a zero-profit contract, and set $r = \rho$. Next substitute $\theta^* = (\alpha/\beta)\mathcal{E}(\theta^*)$ from (5.10). Algebraic manipulation then yields the liquidity bound n^*:

$$\alpha^{n^*} = \frac{\theta^*}{\beta\underline{V}}\frac{y}{1-\beta} \tag{9.12}$$

Now consider the value state process induced in the credit equilibrium. The value state is an affine function of the credit state. The credit state follows the difference equation

$$B_k = y + rB_{k-1} \tag{9.13}$$

The expected value state is an affine function of the credit state:

$$V_k = \overline{A}_0 + \overline{A}_B B_k$$

Solving for B_k and substituting in (9.13) yields a difference equation in the expected value states:

$$V_k = (1-r)\overline{A}_0 + \overline{A}_B y + rV_{k-1}$$

This equation must be shown equivalent to equation (6.1). The coefficient of V_{k-1} is identical to that in (6.1); it remains to demonstrate that

$$(1-r)\overline{A}_0 + \overline{A}_B y = (1-\beta r)\underline{V} \tag{9.14}$$

where $\underline{V}$ is defined in (7.2).

To demonstrate this equality, first simplify the formula for $\overline{A}_0$ in (8.7'):

$$\overline{A}_0 = \beta \overline{A}_B \frac{y}{1-\beta}\frac{1}{\beta(1-r)}(r^{n^*}(1-\beta r) - (1-\beta))$$

Substituting into the left hand side of (9.14) and solving yields

$$\beta \overline{A}_B \frac{y}{(1-\beta)\beta} r^{n^*}(1-\beta r)$$

Now recall that $\beta \overline{A}_B = \theta^*$ and $r = \alpha^{-1}$, use the equality $\theta^* = \beta r \mathcal{E}(\theta^*)$ from (5.10), and the solution of r^{n^*+1} from (9.11), yielding

$$(1-\beta r)\underline{V}$$

which is identical to the right hand side of (9.12).

Next, it must be verified that the minimum value state induced by the minimum credit balance holding in the credit equilibrium is the same as the minimum value state in the contract. Substituting the minimum credit balance from (8.4) into the formula for the expected value state in the credit equilibrium yields

$$\overline{A}_0 + \overline{A}_B \frac{1-r^{n^*+1}}{1-r} y$$

Using the formulae in (8.7-8) and the formula in (9.12) yields after tedious algebra

$$\beta \overline{A}_B \frac{y}{1-\beta}\frac{r^{n^*}}{\beta}$$

Substituting for r^{n^*} from (9.12) yields $\underline{V}$.

This demonstrates that the support of Ψ is identical in the credit equilibrium and in the contract. Clearly the probabilities ψ_k are the same in each since $\theta^* = \beta \overline{A}_B$, and hence (iii) has been shown. ∎

10. A CURRENCY EQUILIBRIUM

I now set out a cash in advance currency equilibrium. I first state the individual's optimization problem in standard form. In standard form, individuals observe the current realization of their shock and choose current consumption and next period currency. They consume up to the current value of depreciated or appreciated real balances chosen in the previous period.

Currency and tickets function identically, as noted in section 2. Each family accumulates tickets, each of which is good for some fixed amount of babysitting time; that value can change over time, yielding a return to ticket holders. Babysitting can only be obtained up to the current value of a family's ticket holdings, in other words, they must obey a ticket (cash) in advance constraint. The equilibrium associated with the tickets is therefore a currency equilibrium. I will now define and characterize the monetary equilibrium, restating results set out in Taub (1988). The monetary equilibrium's correspondence with contract optima is established in the next section.

In recursive form, the individual's maximization problem is

$$V(M,\theta) = \max_{c,M'}\{\theta c + \beta E\left(V(M',\theta')\right)\} \tag{10.1}$$

subject to the constraints:

$$c + M' \leq rM + \tilde{y} \tag{10.2}$$

$$c \geq 0, \tag{10.3}$$

$$c \leq rM, \tag{10.4}$$

$$M' \geq 0, \tag{10.5}$$

where r is the gross rate of return on real balances M. This last constraint is a cash in advance constraint: consumption must not exceed the depreciated value of previous real balances. The following proposition shows that the Bellman equation is solved by an affine value function and policy functions that are defined over fixed sets, just like the grandparents problem.

PROPOSITION 10.1 *Let the gross rate of return on real balances. r. be such that $r\beta < 1$. Then the monetary optimization problem (10.1-4) is solved by policy functions g, h, value function V, and threshold value θ^* defined by*

$$c = g(\theta, M) = \begin{cases} 0 & \theta < \theta^* \\ rM & \theta \geq \theta^* \end{cases} \tag{10.6}$$

$$M' = h(\theta, M) = rM + \tilde{y} - g(\theta, M) \tag{10.7}$$

$$V(\theta, M) = \begin{cases} \underline{V} + \theta^* rM & \theta < \theta^* \\ \underline{V} + \theta rM & \theta \geq \theta^* \end{cases} \tag{10.8}$$

where θ^* is the unique solution to

$$\theta^* = \beta r\{\theta^* F(\theta^*) + \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta)\} = \beta r \mathcal{E}(\theta^*) \tag{10.9}$$

and with $\underline{V}$ defined by

$$\underline{V} = (1-\beta)^{-1}\beta\theta^*\tilde{y} \tag{10.10}$$

PROOF: The result can be verified by direct substitution. Substituting the conjectured form of the value function stated in (10.8) into the right hand side of the recursion (10.1), yields a problem whose optimum is a corner solution, yielding the policies in (10.6-7).

Substituting these policies yields a recursion as follows:

$$\underline{V} + \theta^* rM = \beta(\underline{V} + \mathcal{E}(\theta^{*\prime})(rM + \tilde{y})), \qquad \theta < \theta^*;$$

$$\underline{V} + \theta rM = \theta rM + \beta(\underline{V} + \mathcal{E}(\theta^{*\prime})\tilde{y}), \qquad \theta \geq \theta^*.$$

The coefficients of M in the first of these equations yields (10.9). Substituting the fixed point θ^* into the second yields the solution for $\underline{V}$, in (10.10).

Equation (10.9) is a contraction in θ^*, as demonstrated in the proof of proposition (5.10). ∎

PROPOSITION 10.2 *Let $r\beta < 1$. Then the solution of the Bellman equation recursion in (10.1) is the value function.*

PROOF: See the similar theorem for the credit equilibrium, Proposition D.5. ∎

The individual's optimization problem can now be embedded in a monetary equilibrium.

DEFINITION 10.1 *A monetary equilibrium is a sequence of consumption policy functions $g_t(\cdot,\cdot) : R_+ \times [\underline{\theta}, \overline{\theta}] \to R$, money demand policy functions $h_t(\cdot,\cdot) : R_+ \times [\underline{\theta}, \overline{\theta}] \to R$, prices P_t, tax rates τ_t, money*

growth rates μ_t and gross rates of return on real balances r_t, distribution functions Ψ_t per capita money supplies N_t, N_0 given, such that

(i) the functions g_t and h_t solve the individual's optimization problem (10.1-5);

(ii) the goods market clears so that average consumption equals average endowment:

$$\int g_t(M,\theta)d\Psi_t(M)dF(\theta) = y; \tag{10.11}$$

(iii) the money market clears so that average money held equals the money supply:

$$\int h_t(M,\theta)d\Psi_t(M)dF(\theta) = \int Md\Psi_{t+1}(M) \tag{10.12}$$

(iv) the distribution of real balances has the law of motion determined by the transition rules for money demand:

$$\Psi_{t+1}(M') = \int\int_{\{(M,\theta):M\geq 0,g(M,\theta)\leq M'\}} d\Psi_t(M)dF(\theta) \tag{10.13}$$

(v) net endowment $\tilde{y}$ is the sum of gross endowment y and taxes, τ:

$$\tilde{y}_t = y + \tau_t \tag{10.14}$$

(vi) seignorage equals tax redistributions:

$$\tau_t = \int (1-\mu_t^{-1})Md\Psi_t(M) \tag{10.15}$$

(vii) the price level is fixed by

$$P_t = \left(\int Md\Psi_t(M)\right)^{-1} N_t \tag{10.16}$$

(viii) the rate of return on real balances r_t is the growth rate of purchasing power:

$$r_t = P_{t-1}/P_t \tag{10.17}$$

(ix) the money supply is determined by the growth rule

$$N_t = \mu_t N_{t-1} \tag{10.18}$$

Henceforward I will focus on equilibria that satisfy this definition that are stationary, so that nominal balances, but not real balances, grow over time.

PROPOSITION 10.3 *Let there be a stationary money growth rule $N_t = \mu N_{t-1}$ such that $\beta\mu^{-1} < 1$. Then a stationary monetary equilibrium exists for the economy described in (10.1-5).*

PROOF: Since the equilibrium is stationary, define r from substitution in conditions (10.16-18):

$$r = P_{t-1}/P_t = N_{t-1}/N_t = \mu^{-1}$$

The conditions of Proposition 10.1 are now satisfied, so that g and h as defined in (10.6-7) are optimal for problem (10.1) and the function defined in (10.8) solves (10.1), thus satisfying (i).

Substituting the consumption policy function from (10.6), the market clearing condition (10.11) is

$$\int_{\theta^*}^{\overline{\theta}} [rM]d\Psi(M)dF(\theta) = (1-F(\theta^*))r\int Md\Psi(M) = y \tag{10.19}$$

that is, the fraction of the population that is desperate, $(1 - F(\theta^*))$, consumes available depreciated real balances due to the cash in advance constraint; average consumption must equal the average endowment.

Substituting the policy function for real balances from (10.7) into the money-market clearing condition (10.12) yields

$$\int_{\underline{\theta}}^{\theta^*} [rM + \tilde{y}] d\Psi(M) dF(\theta) + \int_{\theta^*}^{\overline{\theta}} \tilde{y} d\Psi(M) dF(\theta) = \int M d\Psi(M)$$

which has solution

$$\int M d\Psi(M) = \frac{\tilde{y}}{1 - rF(\theta^*)}. \tag{10.20}$$

The average of real balances appears in both (10.19) and (10.20). Eliminating this average from the two equations yields the following expression for $\tilde{y}$:

$$\tilde{y} = \frac{1 - rF(\theta^*)}{(1 - F(\theta^*))r} y. \tag{10.21}$$

The gross rate of return is endogenously fixed in a credit equilibrium, but exogenously determined by the government's inflation policy in a currency equilibrium. Since inflation generates seignorage that is redistributed, net endowment becomes endogenous in the currency equilibrium.

Substituting the optimal policy functions of (10.6-7) into the transition rule for the distribution Ψ of real balances in (10.13) yields

$$\Psi_{t+1}(M') = F(\theta^*)\Psi_t(r^{-1}(M' - \tilde{y})) + 1 - F(\theta^*) \tag{10.22}$$

The first term $F(\theta^*)$, is the probability of nondesperation. Families in that state consume zero and accumulate real balances. The argument of Ψ captures all individuals who transit to real balance holdings M' by this saving process. The second term, $1 - F(\theta^*)$, is the probability of desperation. Families in that state consume all real balance holdings and transit to minimum real balance holdings. (These are positive because current income $\tilde{y}$ cannot be consumed directly and must be sold for real balances due to the cash in advance constraint.)

There are two boundary conditions for Ψ. The first states that $\lim_{M \to \infty} = 1$ since Ψ is a probability distribution. The second is that at the minimum value of real balances, $\tilde{y}$, which is attained when a family becomes desperate and spends all available real balances,

The law of motion for the distribution function has a stationary solution. There is a discrete support,

$$\tilde{y}, r\tilde{y}, \cdots, \sum_{s=0}^{k} \tilde{y}, \cdots$$

reflecting the compounded accumulations of families who are repeatedly nondesperate. This support set can be restated more succinctly as

$$\frac{1 - r^k}{1 - r}\tilde{y}, \qquad k = 1, 2, \cdots$$

The following discrete probabilities solve (10.22) and the associated boundary conditions:

$$\psi_k = (1 - F(\theta^*))F(\theta^*)^{k-1}$$

Thus (iv) is satisfied.

Average real balances can now be calculated. They are

$$\int M d\Psi(M) = \sum_{k=1}^{\infty} (1 - F(\theta^*))F(\theta^*)^{k-1} \frac{1 - r^k}{1 - r} \tilde{y} \tag{10.23}$$

If there is convergence, the right hand side can be solved and the equation reduces to

$$\int M d\Psi(M) = \frac{\tilde{y}}{1 - rF(\theta^*)} \tag{10.24}$$

which is identical to (10.20).

The lump sum redistributions of seignorage (or tax if there is a subsidy to real balances) is determined by solving for the value of seignorage as follows:

$$\tau = (N_t - N_{t-1})/P_t = (1 - \mu^{-1})N_t/P_t$$

Substituting average real balances for N_t/P_t yields (10.15). Simple algebra can be used to verify that (10.15) is in agreement with the solution for τ that results from substituting (10.21) into condition (10.14), yielding

$$\tau = \frac{(1-r)y}{(1 - F(\theta^*))r}.$$

This completes the proof. ∎

One can interpret this in one of two ways. The first is that the monetary authority chooses a tax rate τ, and from that the money growth rate and r are determined by the equilibrium conditions. Alternatively, the monetary authority chooses a money growth rate and associated rate of return, and the tax is then determined by the equilibrium conditions.

11. THE ZERO PROFIT CONTRACT IS A CURRENCY EQUILIBRIUM

I now demonstrate that the ticket system mentioned in section 2 is equivalent to the credit system by showing both to be equivalent to a zero profit grandparents contract. Moreover, the cash in advance constraint of the currency problem can be derived directly from the binding incentive constraint of the contract.

In standard form, individuals observe the current realization of their shock and choose current consumption and next period currency. They consume up to the current value of depreciated or appreciated real balances chosen in the previous period. I show this problem to be equivalent to a problem in which the constraints are stated for every outcome θ and in which policy functions are chosen directly. In this form, the correspondence with the inverse dual can then be established just as in the credit equilibrium: the monetary variables are defined in terms of the dual of the inverse dual of the contract problem.

LEMMA 11.1 *The monetary optimization problem with a cash in advance constraint (previous.1-4) is equivalent to the following problem:*

$$V(M) = \max_{g(\cdot),h(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} (\theta g(\theta) + \beta V(h(\theta)))\, dF(\theta) \tag{11.1}$$

subject to the constraints

$$\int_{\underline{\theta}}^{\overline{\theta}} (g(\theta) + h(\theta))\, dF(\theta) \leq \tilde{y} + rM, \tag{11.2}$$

$$g(\theta) \leq rM, \tag{11.3}$$

$$g(\theta) \geq 0, \tag{11.4}$$

$$h(\theta) \geq 0, \tag{11.5}$$

PROOF: In the problem (11.1-5), policy functions are chosen rather than direct actions, as with the similar proposition 9.2 in the section on credit. The result follows by a straightforward extension of lemma E.1. The details are left to the reader. ∎

PROPOSITION 11.2 *The monetary optimization problem with a cash in advance constraint (11.1-5) is equivalent to the inverse dual of the grandparents' contract, and the binding cash in advance constraint (11.3) of the monetary problem is equivalent to the binding incentive constraint of the grandparents' contract.*

PROOF: To convert the dual of the inverse dual into a currency problem, I will use the same technique of redefining the choice variables as was done to generate the credit problem. First, define $M \equiv \alpha(y + w - (1-\alpha)\overline{W})$. Using the lemma, the family's problem (9.9-10) can now be stated as follows.

$$V(M) = \max_{\gamma_i, M_i'} \{\phi\theta_1\gamma_1 + (1-\phi)\theta_2\gamma_2 + \beta(\underline{V} - W_V^{-1}(\alpha^{-1}\tilde{y}) - W_V^{-1}(\phi M_1' + (1-\phi)M_2'))\} \tag{11.6}$$

subject to

$$\phi\gamma_1 + (1-\phi)\gamma_2 + (\phi M_1' + (1-\phi)M_2') \leq \alpha^{-1}M + \tilde{y} \tag{11.7}$$

$$\gamma_i \leq \alpha^{-1}M, \tag{11.8}$$

$$\gamma_i \geq 0. \tag{11.9}$$

$$M_i' \geq 0. \tag{11.10}$$

The constraint (11.8) arises from substituting the solution for w_1 stated in lemma 9.1 into the binding incentive constraint; it is the precursor of the cash in advance constraint. The cash in advance constraint follows from setting $r = \alpha^{-1}$:

$$\gamma_i \leq rM \tag{11.11}$$

where r is the gross rate of return on real balances.

The following observations can be used to define the net endowment. In the desperate state, $\theta = \theta_2$, the family's problem is solved by choosing maximum consumption and minimum future value, $w_2 = 0$. If the desperate state is then repeated, the current value state is $w = 0$. In the desperate state of a corresponding ticket or currency equilibrium, families cannot use their current endowment for consumption because of the cash in advance constraint. Therefore they put all their net endowment into tickets. Using the definition of tickets with $w = 0$, net endowment is therefore

$$M_2' = \alpha(y - (1-\alpha)\overline{W}) \equiv \tilde{y}.$$

Finally, using the equivalence $W_V^{-1} = -\mathcal{E}(\theta^*)$ yields the problem

$$V(M) = \max_{\gamma_i, M_i'} \{\phi\theta_1\gamma_1 + (1-\phi)\theta_2\gamma_2 + \beta(\underline{V} + \mathcal{E}(\theta^*)(r\tilde{y}) + \mathcal{E}(\theta^*)(\phi M_1' + (1-\phi)M_2'))\} \tag{11.11}$$

subject to

$$\phi\gamma_1 + (1-\phi)\gamma_2 + (\phi M_1' + (1-\varphi)M_2') \leq rM + \tilde{y} \tag{11.12}$$

$$\gamma_i \leq rM, \tag{11.13}$$

$$\gamma_i \geq 0. \tag{11.14}$$

$$M_i' \geq 0. \tag{11.15}$$

Using the solution of the Euler equation for θ^*, (5.10), this problem is by lemma 11.1 identical to that of (10.1-5). ■

The individual's optimization problem can now be embedded in a monetary equilibrium.

PROPOSITION 11.3 *For each α such that $\beta\alpha^{-1} < 1$, there is a zero-profit contract that is equivalent to a monetary equilibrium such that the following conditions hold: (i) $\alpha = r^{-1}$; (ii) the stochastic process of consumption in the contract is identical to the stochastic process of consumption in the monetary equilibrium; (iii) the distribution of states Ψ in the contract is identical to the distribution of credit Ψ in the monetary equilibrium; (iv) the minimum value state $\underline{V}$ of the contract is identical to the minimum value state attainable in the monetary equilibrium as determined by the nonnegativity constraint on nominal balances.*

PROOF: Let $\mu = \alpha^{-1}$. Then by proposition 10.3 there is a monetary equilibrium with $r = \alpha^{-1}$. It remains to be shown that there is a correspondence of the other elements of the equilibrium with the contract.

By proposition 11.2, the monetary optimization problem of the individual is dual to the grandparents problem. The rate of return, r now determines the net endowment, $\tilde{y}$, as demonstrated in proposition 10.3:

$$\tilde{y} = \frac{1 - rF(\theta^*)}{(1 - F(\theta^*))r} y$$

Substitute this value into the expression for $\underline{V}$ in (10.10), yielding

$$\underline{V} = \frac{\beta}{1-\beta}\theta^* \frac{1 - rF(\theta^*)}{(1 - F(\theta^*))r} y$$

Using equation $\theta^* = (\beta/\alpha)\mathcal{E}(\theta^*)$ from (B.10) and letting $r = \rho$, this is easily shown to be equivalent to the expression for $\underline{V}$ in equation (7.2) of the zero-profit contract. ■

This reasoning and proposition 9.2 prove the following proposition.

PROPOSITION 11.4 *Currency and credit are equivalent to a contract that satisfies $E(\gamma) = y$.*

12. ASSET EFFICIENCY

The remaining issue is the degree of efficiency attainable through asset equilibria and the associated rate of return. The efficiency frontier delineated by the grandparents' zero-profit contract has already been found; I simply note the rate of return associated with it. Because of the equivalence of these contracts to asset equilibria, this efficiency bound applies to asset equilibria as well. The efficiency bound requires borrowing constraints to be nontrivially binding, and as I showed in my 1988 paper, generically prevents the optimum quantity of money rule advocated by Friedman (1969) from being implemented.

PROPOSITION 12.1 *(i) If F is nonatomic, the upper bound $\bar{r}$ of the equilibrium gross rate of return satisfies*

$$\bar{r} = F(\theta^*)^{-1},$$

with $\bar{r}$ strictly less than the inverse of the internal discount factor β^{-1}, and $F(\beta\overline{A}_B) < 1$. (ii) The upper bound of welfare is attained at $\bar{r}$. (iii) Insurance in a credit or currency equilibrium is incomplete.

PROOF: (i) This is a corollary of the equivalence of the credit equilibrium and contract, and proposition 7.1. Using the equivalence of currency and credit in proposition 10.1, the same reasoning applies to currency equilibria. ■

The problem is that as the interest rate approaches the internal rate of discount, there is too much attempted saving, not too little. But the resources to reward this saving do not exist in the economy. In monetary equilibria, the attempt to violate the bound on the rate of return encourages excessive holding of real balances, requiring lump sum taxes to finance the associated deflation that exceed income. [12.1] I will next present the intuition underlying this finding.

13. A FOLK THEOREM

A fundamental result in game theory is a "folk theorem" that states that in repeated games, reputational equilibria can be supported for sufficiently large discount factors (Fudenberg and Maskin, 1986). A folk-theorem-like result applies to the present model: the nonexistence of equilibria at high interest rates which is due to adverse selection can be overcome by making the discount factor sufficiently high if the states are discrete. [13.1] As I have modelled it here, the θ-shocks are distributed over a continuum. However,

[12.1] This seems to parallel Bewley's (1983) result that there is an upper bound to the interest rate in currency equilibria with stochastic endowments, which he expresses as individuals being unable to afford to pay their taxes.

[13.1] Rubinstein and Yaari (1983) also found such a folk theorem result for a model of moral hazard in which there is no discounting. Radner (1985) finds a folk theorem in games with discounting in which histories of actions are reviewed, much like the grandparents here implicitly review the histories of consumption of the family.

they can just as easily be modelled over discrete states or a continuum with atoms. It turns out that if the discreteness is coarse enough relative to the discount factor, then complete-insurance equilibria can be sustained. Thus the folk theorem does not apply in its common form if there is a continuum of states. The analogue of the folk theorem here is rather that the completeness of the insurance converges monotonically to complete insurance as the discount factor converges to unity.

PROPOSITION 13.1 *If $F(\cdot)$ has an atom at $\overline{\theta}$, then there is a $\underline{\beta}$ such that for $\underline{\beta} < \beta < 1$ a credit equilibrium with complete insurance exists.*

PROOF: The intuition of the proof is that it is possible to take a nonatomic distribution and compress all the realizations of θ that exceed θ^* into a single point; that point is such that $\overline{\theta} = \theta^*$. First, set $\beta \overline{A}_B = \overline{\theta}$. Set the equilibrium interest rate r equal to $\lim_{\theta \to \overline{\theta}} F(\overline{\theta})^{-1}$. Then choose $\underline{\beta} = \lim_{\theta \to \overline{\theta}} F(\overline{\theta})$. Since the latter quantity is fixed by the distribution F, some β can be found such that the upper bound of the interest rate is $1/\underline{\beta}$. Expected welfare is therefore $\overline{\theta} y/(1-\beta)$, which results from complete insurance. ∎

My 1988 paper presents an example of such an equilibrium. Obviously, the result can be approximated by nonatomic distribution functions.

By loosening the borrowing constraint and making the maximal need state have enough probability, then, complete insurance can be attained. The folk theorem does not apply in the nonatomic case because for any discount factor less than one, it is possible to find a division of the continuum such that the need state has sufficiently small probability that the conditions of proposition 13.1 are not satisfied.

The folk theorem result is connected to the adverse selection concept set out in the classic paper of Rothschild and Stiglitz (1976). I will begin by recasting the Rothschild-Stiglitz model in the stochastic linear utility mold I have been using here. The basic result of Rothschild and Stiglitz is that if the population of high risk types is small enough, no separating (nor pooling) equilibrium will exist. Here, a high risk type will be a family that is currently desperate.

Each family lives two periods and has stochastic utility. The need states are perfectly negatively correlated in that if an family is desperate now it is not desperate in the second period and vice versa, but the current need state is private information. In this sense I am following the modelling strategy of Green and Oh (1989). There are just two need states, $\underline{\theta}$ and $\overline{\theta}$. Each family has lifetime utility

$$\theta c + \beta \theta' c'$$

All families are endowed with W units of time (but the endowment is not state-dependent as it is in Rothschild and Stiglitz's model). The population has two types: fraction λ is desperate in the current period and nondesperate later, and vice versa for the second type which comprises $1-\lambda$ of the population.

It is obviously efficient to separate these groups and give all endowment to the currently desperate families in return for their promise to give their endowment to the opposite group in the second period. If a budget line with fair odds, that is of slope $-\lambda/(1-\lambda)$, is offered to all families, the currently desperate families will consume W/λ and the currently nondesperate families will consume nothing in the current period and $W/(1-\lambda)$ in the second period, as shown in the figure:

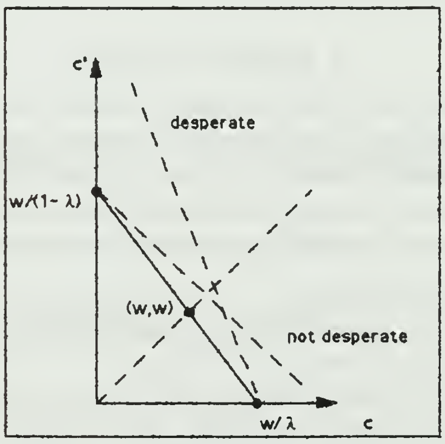

FIGURE 1

The slopes (marginal rates of substitution) of the indifference curves are $-\overline{\theta}/\beta\underline{\theta}$ in the case of the currently desperate, and $-\underline{\theta}/\beta\overline{\theta}$ in the case of the currently nondesperate. As long as they bracket the slope of the feasible budget (the fair odds line of Rothschild and Stiglitz), there will be a separating equilibrium. But if the marginal rate of substitution of the currently nondesperate grows too steep, they will declare themselves desperate and there will be pooling and consequently no insurance. This can happen if the discount factor shrinks too much, if the ratio of the minimum and maximum marginal utilities becomes too extreme, or if the population of currently-desperate families becomes too small.

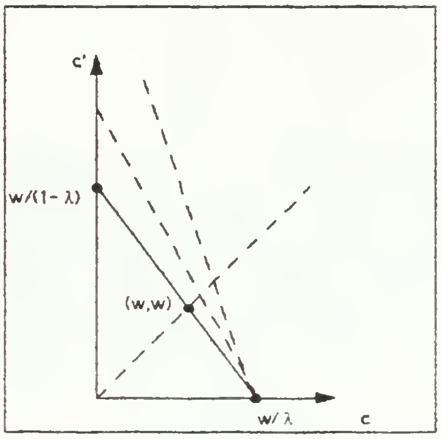

FIGURE 2

Observe that the adverse selection intuition of Rothschild and Stiglitz transfers to this setting. If the proportion of desperate families λ is small, the "high risk" types have more complete insurance. However if λ becomes too small, the non-desperate families do not wish to save because the physical endowment of the desperate families does not finance an interest rate sufficient to make the other families be patient, and there is no equilibrium with insurance.

To what extent does this intuition translate into the infinite horizon framework? With infinitely many periods, the combination of averaging and discounting make the equilibrium essentially like that of the two-period case. With need realizations independent across time, those with high current need are essentially like those families who are desperate in the first period in the two-period setting, and vice versa for those with low current need. In an infinite horizon setting however, there is a possibility of infinite runs of low need or of desperation: $(\underline{\theta}, \underline{\theta}, \ldots)$; $(\overline{\theta}, \overline{\theta}, \ldots)$. This would correspond to having an extra point on a Rothschild-Stiglitz diagram, one to the northeast of the endowment point. In this setting that type would always claim desperation and a separating equilibrium would be impossible. But the law of large numbers combined with discounting essentially rules out such realizations, leaving just the two I analyzed. [13.2] With just those two cases, if desperation is too rare, then λ is too low, and the physical resources needed to finance the return to patience won't exist in the infinite horizon setting either.

Having a continuum of realizations of need corresponds to adding types to the two-period model, and pooling is then inevitable. When there is a continuum of types between $\underline{\theta}$ and $\overline{\theta}$, λ is endogenous. Partitioning the population into those with $\theta < \theta^*$ and $\theta \geq \theta^*$ and making θ^* large, with consumption in the initial period being given to the latter group, will not work if the goal is to reveal who is most desperate.

The nonexistence of high-welfare credit or currency equilibria when there is a continuum of states is thus a kind of adverse selection, corresponding to having "too many" states, resulting in some pooling of high risk or high need types. In a conventional utility setting with stochastic income, wealth can take on infinitely many states, driving marginal utility and utility as well into too many states, leaving high-marginal utility states with a small fraction of the population, so that discounting induces violation of the incentive constraints—pooling—for those close to those states. Counterexamples (such as my own in my 1988 paper)

[13.2] If however the family need shocks were serially correlated, such northeast points would become more probable, and in the limit as serial correlation approached a martingale or random walk, could not be ruled out. Intuitively, insurance would become impossible since no separation would be feasible. This is exactly what emerges in my 1990 paper.

depend on having discrete states and large discount factors so as to effect separation. Institutions that restrict states so as to effect separation are thus likely to be the only kind that achieve separation. As in all equilibria of this type, no institution can be expected to overcome the "inefficiency" that accompanies the adverse selection pressure—what Rothschild and Stiglitz call the information externality.

14. AN ANTI-FOLK THEOREM

The folk theorem of the previous section states that if families are very patient, complete or almost complete insurance can be attained. However if families are less patient, the upper bound of welfare is defined by the *lower* bound of α, the grandparents' discount factor. As the grandparents become more patient, or equivalently as the asset rate of return falls, welfare falls.

The grandparents' contract presumes full commitment by both the grandparents and the family. Similarly, asset equilibria presume commitment by families to continuing participation. In a representative-agent model of asset equilibria, commitment is not an issue because asset equilibria necessarily reflect all alternatives; interest rates are shadow values of immutable allocations. In the babysitting cooperative, however, there are alternatives to remaining in the contract. [14.1] There is a minimum value state, and families may prefer to revert to autarky from that state rather than remain with the contract if it offers a higher expected utility than continued participation. I will call a contract *immune from defection* if reverting to autarky is never preferred. [14.2] Equilibria that are not immune from defection should not be observable. It is a nontrivial restriction on the set of contracts as I show below.

It is simple to check whether a contract is immune from defection: it must be the case that the minimum value state exceeds the average value under autarky. [14.3] Using the formula for the minimum value state (7.2), a feasible contract is immune from defection if

$$\frac{1-\rho F(\theta^*)}{1-F(\theta^*)}\mathcal{E}(\theta^*)\frac{y}{1-\beta} \geq E(\theta)\frac{y}{1-\beta}. \tag{14.1}$$

Define $\underline{\alpha}$ such that the solution to (5.10) is $F(\theta^*) = \underline{\alpha}$. This is the infimum of the α's that sustain feasible equilibria, and is associated with the supremum of family welfare. The following proposition states that feasible equilibria such that welfare is close to the supremum are not immune from defection.

PROPOSITION 14.1 *There exists $\epsilon > 0$ such that for all $\alpha \in (\underline{\alpha}, \underline{\alpha}+\epsilon)$, the associated feasible contracts are not immune from defection.*

PROOF: Since in the grandparents' contract $\rho = \alpha^{-1}$, and $\lim_{\alpha\to\underline{\alpha}} F(\theta^*) = \rho^{-1}$,

$$\lim_{\alpha\downarrow\underline{\alpha}} \frac{1-\rho F(\theta^*)}{1-F(\theta^*)}\mathcal{E}(\theta^*) = 0$$

which violates the immunity condition (14.1). ∎

Feasible contracts that are immune from defection exist and are associated with patient grandparents but low asset rates of return and low average welfare.

PROPOSITION 14.2 *There exists α such that the associated feasible contracts are immune from defection.*

[14.1] Gale (1982. p. 234) called this the trust problem in the context of assets.

[14.2] This idea was examined by Thomas and Worrall in the context of a contract in which the discount factor was the same for both principal and agent. This is a more primitive concept than renegotiation proofness. Families that reverted to autarky could later apply for readmittance; I assume that a grim strategy is used on those families. This delineates the largest set of equilibria that are sustainable against any punishment strategy. An alternative approach would be to incorporate a sequential rationality constraint in the contract objective; this was recently done in a planner model by Marimon and Marcet (1992).

[14.3] I analyze only the case in which reversion to autarky takes place after the contract is executed in the current period.

PROOF: Since $\beta < 1$, $\lim_{\alpha\to 1} \mathcal{E}(\theta^*) > E(\theta)$. Using the Euler condition (6.10) and the fact that $\rho = \alpha^{-1}$,

$$\lim_{\alpha\uparrow 1} \frac{1-\rho F(\theta^*)}{1-F(\theta^*)}\mathcal{E}(\theta^*) > E(\theta)$$

which satisfies the immunity condition (14.1). ■

Even if the defection constraint is exactly binding, there will be a distribution of family value states with only the minimum one, $\underline{V}$, equal to the value of autarky. The mean family value state will therefore exceed the minimum, and in this sense the contract still strictly dominates autarky.

The results of this section are in agreement with those of Haller (1989) who examines a two-person repeated game with an insurance aspect; the results are also in agreement with the more general finding that dynamically consistent asset equilibria seem to require low or negative rates of return, such as in Atkeson (1991), Manuelli (1985), and Kehoe and Levine (1990). The high value of α translates into low (or even negative) interest rates being necessary for asset equilibria being immune from defection. While high inflation and strict credit restrictions have a mundane explanation in countries with weak or rapacious governments, they are harder to explain in democratic societies when the efficiency of low inflation and unrestricted markets is so widely understood. The theory presented here predicts that inflation and credit restrictions might be necessary to sustain immunity from defection.

15. CONCLUSION

The grandparents' discount factor is exogenous, and the minimum value state permitted is also exogenous. When zero profit is in addition imposed, only one of these parameters can be exogenous, the other becoming endogenous. This same pattern is repeated in the asset equilibria: in the credit equilibrium, the borrowing constraint is exogenous, and the rate of return endogenous; in the currency equilibrium, the rate of return is exogenous, and the net of tax endowment endogenous.

The equilibria can be characterized by discount factors. With grandparents who discount the future heavily, efficiency can be approached within the confines of the incentive constraints. When the cooperative attempts to duplicate a grandparents' contract, the contract must be resource feasible, and due to the concommitant adverse selection the maximum sustainable rate of return falls, along with efficiency. When immunity from defection is also required, the maximum sustainable interest rate is even lower, as is efficiency. Immunity from defection also necessitates stationarity of the distribution of wealth and consumption. Only the latter equilibria should be empirically observable; they explain why inefficient inflation rates and credit restrictions seem to be the rule.

While the technical framework here is narrow, the findings are striking in light of the fact that assets have a prominent role in economic growth and business cycles. My results suggest that cash in advance models are the correct way to model currency, but that heterogeneous-agent models are needed for this motivation to have analytical meaning. It may therefore be possible to advance our understanding of growth and business cycles with an extension of the contract approach.

January 31, 1993

APPENDIX

A. MONOTONICITY INDUCED BY THE CONSTRAINTS

In this appendix I show that the grandparents' policy functions have a monotonicity property which in turn greatly simplifies the sets over which family value states stochastically evolve. The strategy of this section will be to show that there are two types of sets of values of θ, A and B. Within each set there is a fixed strategy, and across the sets the incentive constraints hold in a particular way. Then I show that the sets necessarily have a monotonic relationship with each other. Finally, I show that no other sets of admissible θ exist.

PROPOSITION A.1 *For $\theta > s$, (i) $\gamma(\theta, V) \geq \gamma(s, V)$, and (ii) $V'(\theta, V) \leq V'(s, V)$.*

PROOF: (i) The incentive constraints corresponding to θ and s are

$$\theta\gamma(\theta, V) + \beta V'(\theta, V) \geq \theta\gamma(s, V) + \beta V'(s, V), \tag{A.1}$$

$$s\gamma(s, V) + \beta V'(s, V) \geq s\gamma(\theta, V) + \beta V'(\theta, V) \tag{A.2}$$

Adding the two conditions yields

$$(\theta - s)(\gamma(\theta) - \gamma(s)) \geq 0. \tag{A.3}$$

Since $\theta \geq s$, $(\gamma(\theta) - \gamma(s)) \geq 0$.

(ii) Using the result from (i), (A.2) can be written

$$\beta V'(s, V) - \beta V'(\theta, V) > s(\gamma(\theta, V) - \gamma(s, V)) > 0 \tag{A.4 - 5}$$

which proves the result. ∎

PROPOSITION A.2 *For the set A such that for $\theta \in A, \gamma(\theta, V) = 0$, then $V'(\theta, V)$ is constant.*

PROOF: Substituting $\gamma(\theta, V)$ into (A.1-2) and supposing that both $\theta \in A$ and $s \in A$, the incentive constraints (A.1) and (A.2) are symmetric, and therefore $V'(\theta, V) = V'(s, V)$ necessarily holds. ∎

PROPOSITION A.3 *If there exists a set B of positive F-measure such that for $\theta \in B$, $V'(\theta, V) = \underline{V}$, then $\gamma(\theta, V)$ is constant with respect to θ.*

PROOF: Substitute $\underline{V}$ into (A.1-2). Then in order for both inequalities to be satisfied, symmetry requires that both constraints hold with equality, and the result follows. ∎

PROPOSITION A.4 *If $s \in A$ so that $\gamma(s, V) = 0$ and $\theta \in B$ so that $V'(\theta, V) = \underline{V}$, then $s \leq \theta$.*

PROOF: All incentive constraints must hold with equality within the sets A and B due to symmetry. The remaining incentive constraints take the form

$$\theta\gamma(\theta, V) + \beta\underline{V} \geq \beta V'(s, V), \tag{A.7}$$

$$\beta V'(s, V) \geq s\gamma(\theta, V) + \beta\underline{V} \tag{A.8}$$

The right hand side of (A.7) is identical to the left hand side of (A.8), and so the combined inequality can be stated

$$\theta\gamma(\theta, V) + \beta\underline{V} \geq s\gamma(\theta, V) + \beta\underline{V}$$

which requires $\theta \geq s$. ∎

PROPOSITION A.5 *If $\gamma > 0$ for some set of positive F-measure, then the recursion constraint is binding.*

PROOF: Suppose the contrary for the optimal functions γ and V'. Then choose a constant ϵ such that $\gamma(\theta, \theta) \geq \epsilon$, for all θ on some set A. Because ϵ is constant, the incentive constraints will not be violated within this set since $\theta\epsilon$ is subtracted from both sides of each incentive constraint. Outside the set, $\gamma < \epsilon$. The incentive constraints across the two sets must mesh. Suppose $\theta \in A$ and $s \notin A$. The pair of constraints is

$$\theta\gamma(\theta) + \beta V'(\theta) \geq \theta\gamma(s) + \beta V'(s)$$

$$s\gamma(s) + \beta V'(s) \geq s\gamma(\theta) + \beta V'(\theta).$$

Since by assumption $\gamma(s) < \epsilon$, then $\gamma(s) < \gamma(\theta)$ as well. By proposition A.1, $\theta > s$. Cascading the inequalities leaves

$$\theta\gamma(\theta) \geq s\gamma(\theta)$$

and therefore

$$\theta(\gamma(\theta) - \epsilon) \geq s(\gamma(\theta) - \epsilon).$$

Substituting $\theta(\gamma(\theta) - \epsilon)$ into the recursion constraint (4.2) leaves the recursion constraint satisfied. The maximum of the problem (1.1) is therefore raised by

$$\epsilon \int_A dF$$

which contradicts the optimality of γ. ∎

PROPOSITION A.6 *There is a single θ^* dividing the sets A and B, so $A \cup B = [\underline{\theta}, \overline{\theta}]$.*

PROOF: Suppose the contrary, so that there is a set X such that both $\gamma_x > 0$ and $V_x > \underline{V}$. The upper bound of the set A is s^*, and the lower bound of set B is t^*, so that by monotonicity, $s^* < t^*$. Again by monotonicity, $0 < \gamma(x, V) < \gamma(t, V)$ and $\underline{V} < V'(x, V) < V'(s, V)$.

The incentive constraints must hold at s^*, the lower bound of X and the upper bound of A:

$$\beta\overline{V} \geq s^*\gamma_x + \beta V_x \geq \beta\overline{V}$$

and therefore all hold with equality. Also,

$$\beta\overline{V} \geq s^*\overline{\gamma} + \beta\underline{V}.$$

The incentive constraints must also hold at t^*, the upper bound of X and the lower bound of A:

$$t^*\overline{\gamma} + \beta\underline{V} \geq t^*\gamma_x + \beta V_x \geq \beta\overline{V}.$$

Since $t^* > s^*$, both inequalities are strict.

It is therefore possible to raise and lower $\gamma(x)$ and $V'(x)$ by small amounts without violating the incentive constraints. For a localized change at s^* (where the incentive constraints are binding) which reduces $\gamma(s^* + \epsilon)$ and increases $V'(s^* + \epsilon)$, $s^* + \epsilon \in X$, this requires

$$\frac{d\gamma}{dV'} = -\frac{\beta}{s^*}$$

which does not alter the recursion constraint. The return to the grandparents is reduced by $dF(s^*)d\gamma$, while the functional $E[W(V')]$ is increased due to the change in V'. It is optimal to undertake the decrease in γ if the the objective decreases with respect to the changes of γ and V'. otherwise it is optimal to increase γ_x and reduce V_x in like manner. The limits on this change are the nonnegativity and minimum value constraints (4.4-5). ∎

Thus there is a cutoff value of θ, θ^*, such that below that point a single policy is followed by the grandparents and vice versa:

$$\gamma(\theta) = \begin{cases} 0, & \underline{\theta} \leq \theta < \theta^* \\ \overline{\gamma}, & \theta^* \leq \theta < \overline{\theta} \end{cases} \tag{A.9}$$

$$V'(\theta) = \begin{cases} \overline{V}, & \underline{\theta} \leq \theta < \theta^* \\ \underline{V}, & \theta^* \leq \theta < \overline{\theta} \end{cases} \tag{A.10}$$

The incentive constraints have the form

$$\begin{aligned} \beta\overline{V} &\geq \theta\overline{\gamma} + \beta\underline{V} \qquad && \underline{\theta} \leq \theta < \theta^*, \\ \theta\overline{\gamma} + \beta\underline{V} &\geq \beta\overline{V} && \theta^* \leq \theta < \overline{\theta}. \end{aligned}$$

Both constraints will be satisfied if and only if the hunger threshold θ^* satisfies the equation

$$\theta^*\overline{\gamma} + \beta\underline{V} = \beta\overline{V}. \tag{A.11}$$

The grandparents' problem is now reduced to finding the characteristics of solutions with the pairs of threshold values $\overline{\gamma}$ and $\overline{V}$ and finding the optimal hunger threshold, θ^*. It should be emphasized that θ^* need not be in $[\underline{\theta}, \overline{\theta}]$. If it is not, the equilibrium is degenerate. The condition necessary and sufficient for $\theta^* \in [\underline{\theta}, \overline{\theta}]$ is presented in section 6.

B. SOLUTION OF THE GRANDPARENTS' PROBLEM

With the finding that policies necessarily have the threshold form, the grandparents' maximum problem can be restated in these terms:

$$\max_{\theta^*,\overline{\gamma},\overline{V}} \{y - (1 - F(\theta^*))\overline{\gamma} + \alpha F(\theta^*)W(\overline{V}) + \alpha(1 - F(\theta^*))W(\underline{V})\} \tag{B.1}$$

subject to

$$V = \int_{\theta^*}^{\overline{\theta}} \theta\overline{\gamma}dF(\theta) + \beta F(\theta^*)\overline{V} + \beta(1 - F(\theta^*))\underline{V} \tag{B.2}$$

and the binding incentive constraint

$$-\theta^*\overline{\gamma} + \beta(\overline{V} - \underline{V}) = 0. \tag{B.3}$$

For notational convenience, I now define the quantities $\Phi(\theta^*) \equiv \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta)$ and $\mathcal{E}(\theta^*) \equiv F(\theta^*)\theta^* + \Phi(\theta^*)$. (The notation $\mathcal{E}$ is purposely meant to evoke statistical expectation: if F is Bernoulli, and θ^* is between the two realizations, then $\mathcal{E} = E(\theta)$.) In matrix form the two constraints then take the form

$$\begin{pmatrix} \overline{\gamma} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} \Phi(\theta^*) & \beta F(\theta^*) \\ -\theta^* & \beta \end{pmatrix}^{-1} \begin{pmatrix} V - \beta(1 - F(\theta^*))\underline{V} \\ \beta\underline{V} \end{pmatrix}. \tag{B.4}$$

with solution

$$\overline{V} = \left(1 - \beta\frac{\theta^*}{\beta\mathcal{E}(\theta^*)}\right)\underline{V} + \frac{\theta^*}{\beta\mathcal{E}(\theta^*)}V, \tag{B.5}$$

$$\overline{\gamma} = \frac{1}{\mathcal{E}(\theta^*)}(V - \beta\underline{V}). \tag{B.6}$$

Because they are affine in the family's current value state, V, these solutions will be used to characterize the growth conditions that limit equilibria.

The problem of finding the optimal threshold value of θ^* can be considered separately from that of finding the optimal functions γ and V'. In this endeavor, posit that W is affine in the current value state: $W(V) = W_0 + W_V V$. I will show that this function solves the dynamic programming recursion; in appendix C I show that it is the grandparents' value function. Using the intuition that W is decreasing in V, the solutions of the maximum problem can be stated easily: consumption will be zero if the family signals nondesperation, and will be at a maximum if desperation is signalled. Thus in the case of desperation, the family's value state will revert to its minimum. In the case of nondesperation, the family's value state grows according to (B.5).

Substituting the optimal values into the value function recursion yields solutions for the coefficients W_0 and W_V. Equating coefficients of powers of V, the solutions are

$$W_0 = \frac{1}{1-\alpha}\left(y - (\beta - \alpha)\frac{\beta(1 - F(\theta^*))}{\beta\mathcal{E}(\theta^*) - \alpha F(\theta^*)\theta^*}\underline{V}\right) \tag{B.7}$$

$$W_V = -\frac{\beta(1 - F(\theta^*))}{\beta\mathcal{E}(\theta^*) - \alpha F(\theta^*)\theta^*}. \tag{B.8}$$

Observe that the marginal value of the state, W_V, depends only on the stochastic structure of the taste shocks, not on the current state.

The optimal need threshold. Invoking the envelope condition, I use the linear programming solutions that hold $F(\theta^*)$ fixed and focus on the problem of choosing θ^*. The first order condition for this problem is

$$\overline{\gamma}F'(\theta^*) + \alpha F'(\theta^*)(W(\overline{V}) - W(\underline{V})) = 0.$$

Dividing out $F'(\theta^*)$ yields

$$\overline{\gamma} = \alpha W'(V)(\underline{V} - \overline{V}) \tag{B.9}$$

using the fact that W is affine. Substituting the solution for $W'(V)$ and for $\underline{V}$ and $\overline{V}$ yields the Euler equation in θ^*,

$$\theta^* = \frac{\beta}{\alpha}\mathcal{E}(\theta^*). \tag{B.10}$$

PROOF: (of Proposition 5.1): The following properties are easily verified for $\mathcal{E}(\theta)$: (i) $\mathcal{E}(\underline{\theta}) = E(\theta)$; (ii) $\mathcal{E}(\overline{\theta}) = \overline{\theta}$; (iii) $\mathcal{E}'(\theta) = F(\theta)$. These conditions, combined with $\alpha > \beta$, imply $\beta/\alpha\mathcal{E}(\underline{\theta}) > 0$, $\beta/\alpha\mathcal{E}(\overline{\theta}) < \overline{\theta}$, and $\beta/\alpha\mathcal{E}'(\theta) \leq \beta/\alpha < 1$. Therefore there is a unique interior crossing point. ∎

COROLLARY B.2 $W_V = -\mathcal{E}(\theta^*)^{-1}$.

PROOF: Substitute (B.10) into (B.8). ∎

C. THE AFFINE FUNCTION IS THE GRANDPARENTS' VALUE FUNCTIONAL

The fixed point of the grandparents' dynamic programming recursion is affine, and is therefore also unbounded and is potentially not the value function. This appendix demonstrates that the affine solution is nevertheless the value function.

The return "function" on the right hand side of the grandparents' recursion is an expected value—a functional; this reflects the grandparents' lack of knowledge of the current realization of the taste shock, θ. The grandparents choose functions, not values of the feasible control. Because the return functional requires the choice of a function, the exercise is like deterministic dynamic programming in spirit, and the methods developed for deterministic dynamic programming can be applied. In particular, I show that all feasible plans—sequences of feasible functions—lead to nonpositive discounted returns, and use the finding analogously with Assumption 2 of Stokey and Lucas's discussion of deterministic dynamic programming (Stokey and Lucas, p. 68). The analogue of their Theorem 4.3 can then be developed.

LEMMA C.1 *Let $\alpha < 1$ and $\beta\alpha^{-1} < 1$. Then for every feasible policy, the affine function W that solves the grandparents' dynamic programming recursion satisfies*

$$\lim_{T\to\infty} \alpha^T \int_{\underline{\theta}}^{\overline{\theta}} W(V^T)dF(\theta) \leq 0$$

PROOF: Since consumption is restricted to be positive, $V \geq \underline{V} \geq 0$ is necessary. Also, by corollary B.2, the marginal value of the state W_V is negative. Therefore,

$$\lim_{T\to\infty} \alpha^T \int_{\underline{\theta}}^{\overline{\theta}} W(V^T(V,\theta))dF(\theta) \leq \lim \alpha^T(W_0 + W_V\underline{V}) = 0.$$

It should be noted that it is feasible for the inequality to be strict. ∎

PROPOSITION C.2 *The affine function W found to satisfy the dynamic programming recursion is the value function.*

Since it is easy to find that the affine function satisfies the dynamic programming recursion, this proposition says that the affine function is in fact the value function.

PROOF: The proof parallels the proof of Theorem 4.3 of Stokey and Lucas (p. 72). There are two main steps. The first is to show that the affine solution weakly dominates all feasible solutions of the optimization problem. The second is to show that the affine solution is weakly dominated by the supremum of the solutions.

The first step begins by using the optimality of the solution of the Bellman equation:

$$W[V^0] \geq \int_{\underline{\theta}}^{\overline{\theta}} \left((y - \gamma^1(V^0,\theta_1)) + \alpha W[V^1(V^0,\theta_1)]\right) dF(\theta_1)$$

where V^1 and γ^1 are feasible functions conditional on the initial function V^0. Iteration yields

$$\geq \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^1(V^0, \theta_1))dF(\theta_1) + \alpha^2 \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^2(V^1(V^0, \theta_1), \theta_2))dF(\theta_1)dF(\theta_2)$$

$$+ \cdots + \alpha^T \int_{\underline{\theta}}^{\overline{\theta}} \cdots \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} W[V^T]dF(\theta_1)dF(\theta_2) \cdots dF(\theta_T)$$

where the arguments of V^T have been omitted for brevity. Taking the limit on the right hand side, the last term is nonpositive by lemma C.1. Since the feasible sequences of functions include the optimal one, the affine solution weakly dominates the optimal sequence.

For the second step, for ϵ arbitrarily small, define a sequence $\{\delta_n\}$ such that the sum $\sum_{n=1}^{\infty} \alpha^{n-1}\delta_n$ is less than or equal to $\epsilon/2$. Choose the sequence of consumption policy functions γ^k and continuation value functions V^k to be optimal, so that by (5.10), $\alpha\rho = 1$. Clearly this is a feasible sequence of policy functions.

For notational brevity define $\overline{V}_k \equiv (1 - \rho^k)V^s + \rho^k \underline{V}$ as in equation (6.5).

The following inequality now holds because of the optimality of (γ^k, V^k):

$$W[V^0] \leq \int_{\underline{\theta}}^{\overline{\theta}} \left((y - \gamma^1(V^0, \theta_1)) + \alpha W[V^1(V^0, \theta_1)]\right) dF(\theta_1) + \delta_1$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} ((y - \gamma^1(V^0, \theta_1))dF(\theta_1) + \alpha W[F(\theta^*)\underline{V} + (1 - F(\theta^*))\overline{V}_1)] + \delta_1$$

The last equality follows because W is affine. Similarly,

$$W[V^1] \leq \int_{\underline{\theta}}^{\overline{\theta}} \left((y - \gamma^2(V^1, \theta_2)) + \alpha W[V^2(V^1, \theta_2)]\right) dF(\theta_2) + \delta_2$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^2(V^1, \theta_2))dF(\theta_2) + \alpha W[F(\theta^*)\underline{V} + (1 - F(\theta^*))\overline{V}_2] + \delta_2$$

Therefore

$$W[V^0] \leq \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^1(V^0, \theta_1)) + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^2(V^1(V^0, \theta_1), \theta_2))dF(\theta_2)dF(\theta_1)$$

$$+\alpha^2 W[F(\theta^*)^2\underline{V} + F(\theta^*)(1 - F(\theta^*))\overline{V}_1 + (1 - F(\theta^*))^2\overline{V}_2] + \delta_1 + \delta_2$$

and so on.

Iterating these inequalities yields

$$W[V^0] \leq \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^1(V^0, \theta_1))dF(\theta_1) + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (y - \gamma^2(V^1(V^0, \theta_1)\theta_2))dF(\theta_2)dF(\theta_1)$$

$$+ \cdots + \alpha^T W[F(\theta^*)^T\underline{V} + F(\theta^*)^{T-1}(1 - F(\theta^*))\overline{V}_1 + \cdots + (1 - F(\theta^*))^T\overline{V}_T] + \epsilon/2.$$

The partial sums in the last part of this expression are weighted sums of constants multiplied by α^T and of the following two elements:

$$\alpha^T \frac{(1 - F(\theta^*))^{T+1} - F(\theta^*)^{T+1}}{1 - 2F(\theta^*)}$$

and

$$\alpha^T \frac{(1 - F(\theta^*)\rho)^{T+1} - (F(\theta^*)\rho)^{T+1}}{1 - 2F(\theta^*)}$$

The first expression converges to zero since $\alpha F(\theta^*) < 1$. The second expression converges to zero since $\alpha\rho = 1$ and $F(\theta^*) < 1$, $(1 - F(\theta^*) < 1$, and the last element of the inequality tends to zero in the limit. Since this reasoning holds for arbitrary positive ϵ, the affine solution is weakly dominated by a feasible sequence of policy functions. ■

D. THE AFFINE FIXED POINT OF THE MAPPING IS THE VALUE FUNCTION FOR INDIVIDUALS

This appendix is devoted to showing that when a growth condition is satisfied for families, that it is legitimate to treat the fixed point that is the solution to the dynamic programming problem in the space of affine functions as the unique value function, just as intuition suggests. The exercise is a mechanical one: I simply show that assumptions 9.1 and 9.2 of Stokey and Lucas (1989) are satisfied for appropriate ranges of the interest rate, borrowing limit, and discount factor and apply their Theorem 9.2.

I also show in this section that when these parameter bounds are violated, the value is infinite, and the associated optimal policy calls for behavior that is infeasible in the aggregate, again as intuition suggests.

I will now draw the notational correspondence with Stokey and Lucas (1989, p. 241). First, the stochastic state space is $Z = \{\theta \in [\underline{\theta}, \overline{\theta}]\}$. The endogenous state is $x = B$, with $B \in [\underline{B}, \infty]$ with the Borel measure applied to the latter. Thus the state space is the measurable product space $(S, \mathcal{S}) = ([\underline{\theta}, \overline{\theta}] \times [\underline{B}, \infty), \mathcal{T} \times \mathcal{B})$.

The set of feasible choices is given by the budget constraint, the liquidity constraint and the nonnegativity constraint on consumption:

$$\Gamma(\theta, B) \equiv \{B' : B' \in [\underline{B}, y + rB]\},$$

so $\Gamma : [\underline{\theta}, \overline{\theta}] \times [\underline{B}, \infty) \to [\underline{B}, \infty)$, and $B' \in \Gamma(\theta, B)$.

The graph of Γ is

$$A = \{(B, \theta, B') \in [\underline{B}, \infty) \times [\underline{B}, \infty) \times [\underline{\theta}, \overline{\theta}] : B' \in \Gamma(B, \theta)\}$$

The return function is:

$$F(B, B', \theta) = \theta(y + rB - B').$$

I now show that Assumption 9.1 of Stokey and Lucas is satisfied.

LEMMA D.1 *(i) Γ is nonempty. (ii) The graph of Γ is $\mathcal{B} \times \mathcal{B} \times \mathcal{T}$-measurable. (iii) Γ has a measurable selection; that is, there is a measurable function $h : S \to X$ such that $H(s) \in \Gamma(s)$, all $s \in S$.*

PROOF: (i) Since $\underline{B}$ is always feasible, Γ is nonempty. (ii) Since Γ is not dependent on the taste shocks, the graph of Γ can be written more simply as

$$A = \{(B, B') \in [\underline{B}, \infty) \times [\underline{B}, \infty) : B' \in \Gamma(B)\}.$$

Thus the graph is the closed, and so is in the product measure. (iii) Clearly Γ is a closed correspondence since it takes values that are closed intervals. Also its domain, $[\underline{\theta}, \overline{\theta}] \times [\underline{B}, \infty)$ is a subset of R^2. This satisfies the conditions of Theorem 7.6 of Stokey and Lucas (p. 184). ∎

Next I show that Assumption 9.2 of Stokey and Lucas is satisfied.

ASSUMPTION D.2 *There is some $n^* > -\infty$ such that*

$$\underline{B} \geq -\frac{1 - r^{n^*+1}}{r} y.$$

Define the sigma algebra of the graph, $\mathcal{A}$, as the extension of the product of the sets $\mathcal{T} \times \mathcal{B} \times \mathcal{B}$ (see Taylor, p.135), and take the product measure.

LEMMA D.3 *(i) $F : A \to R$ is $\mathcal{A}$-measurable and (ii) $F \geq 0$.*

PROOF: (i) One need only note that the components of F are measurable. That is, θ is a $\mathcal{T}$-measurable function on $[\underline{\theta}, \overline{\theta}]$, and $(y + rB - B')$ is the sum of monotone (and hence $\mathcal{B}$-measurable) functions on B. The measure of the product $\theta(y + rB - B')$ is the product of the measures. (ii) Trivial by the definition of F and the restriction of Assumption D.2. ∎

Now I make an assumption that amounts to a growth condition. The discounting of returns must be strong enough to assure convergence of the discounted returns even if the most extreme realizations occur.

The most extreme realizations are of maximum hunger, and the maximum return occurs if all resources are put into consumption at those extreme realizations; if zero consumption occurs in anticipation of such extreme realizations, the accumulated assets can be cashed in when maximum hunger is attained. Thus:

ASSUMPTION D.4 $\beta r < 1$.

With this assumption discounted utility is bounded:

$$\lim_{t\to\infty} \beta^t \frac{r^{t+1}-1}{r}\overline{\theta} y = [(\beta r)^t - \beta^t]\frac{\overline{\theta} y}{r}$$

PROPOSITION D.5 *The function V that satisfies the dynamic programming recursion is the value function.*

PROOF: Since V, the function that satisfies the dynamic programming recursion, is affine in the state, B, it suffices to show that the argument of V satisfies the growth condition in Assumption D.4. This means the hypothesis given in equation (7) of Theorem 9.2 of Stokey and Lucas is satisfied. Also lemmata D.1 and D.3 show that the conditions of the theorem are satisfied. ∎

PROPOSITION D.6 *If the growth condition is violated, no value or policy function exists.*

PROOF: (i) If the growth condition is not satisfied, an individual could follow a strategy of waiting to consume until his hunger were extremely high. Suppose this is the policy function. The discounted value of such a strategy is then infinite. However such an individual would never consume; even if faced with a very high hunger state that exceeded his minimum hunger threshold, he could gain even more in discounted terms by delaying consumption. (ii) Suppose the opposite strategy: delaying consumption forever. This cannot be optimal because it yields zero discounted utility. Consuming at any time attains positive discounted utility and therefore dominates the strategy. The optimal policy function and value function therefore do not exist. ∎

The point of this result is that an equilibrium cannot be sustained if the interest rate is too high; the technical nature of the breakdown is that any conjectured equilibrium that had a high interest rate would not satisfy individual optimization, since no optimum exists.

E. CREDIT EQUILIBRIA

PROOF: [Of Corollary 8.3]: From (8.8) it is apparent that $\overline{A}_B$ is an increasing function of the interest rate, r, so $dF/dr \geq 0$. Since F is an increasing function of $\overline{A}_B$, differentiating the right hand side of (8.14) yields

$$-\left(\frac{F}{1-F} + \frac{(r-1)(dF/dr)}{(1-F)^2}\right) \tag{E.1}$$

which is negative. Therefore, for a given $n^* < 0$, both the left and right hand sides of (8.14) are decreasing in r, and are equal for $r = 1$. The derivative of the left hand side of (8.14) with respect to r at $r = 1$ is $1 + n^*$, which is negative for $n^* \leq -1$. The derivative of the right hand side of (8.14) with respect to r at $r = 1$ is $-F(\beta\overline{A})/(1 - F(\beta\overline{A}))$, which is also negative. As r approaches $1/\beta$, $F(\beta\overline{A}) \to 1$, so the right hand side of (8.14) approaches $-\infty$, but the left hand side remains positive. The derivative of the left hand side at $r = 1$ can be made arbitrarily negative by decreasing n^*, leaving the derivative of the right hand side unaffected. Therefore there must be some interior crossing point such that $1 < r < 1/\beta$ in addition to the intersection at $r = 1$. However at $r = 1$ (8.13) is undefined. ∎

The following lemma shows that when utility is linear there is no difference between a standard stochastic dynamic programming problem in which policies are chosen in response to current realized states, with policy functions then defined by the policies over all possible realizations (as in the family's credit decisions), and one in which the current return function is an average of returns over all states and policy functions are chosen before the realization of the state (as in the grandparents' contract problem).

LEMMA E.1 *Let $V(\cdot,\theta)$ be weakly increasing in its first argument. Define*

$$V^* = \max_{c(\cdot),B'(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} (\theta c(\theta) + \beta V(B'(\theta),\theta))\, dF(\theta) \tag{$*$}$$

subject to the constraints

$$\int_{\underline{\theta}}^{\overline{\theta}} (c(\theta) + B'(\theta))\, dF(\theta) \le y + rB,$$

$$c \ge 0,$$

$$B' \ge \underline{B},$$

and

$$(**) \qquad V^{**}(\theta) = \max_{c, B'} \left\{ \theta c + \int_{\underline{\theta}}^{\overline{\theta}} \beta V(B', \theta') dF(\theta') \right\}$$

subject to

$$c + B' \le y + rB,$$

$$c \ge 0,$$

$$B' \ge \underline{B}.$$

Then

$$V^* = \int_{\underline{\theta}}^{\overline{\theta}} V^{**}(\theta) dF(\theta).$$

PROOF: ($\geq$) Averaging the budget constraint in problem (*) means that all policies feasible under the budget constraints in (**) are also feasible under (*). Therefore the optimal policy function defined by (**) is feasible for (*). Since V^* is an optimum,

$$V^* \ge \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta c^{**}(\theta) + \beta V(B'^{**}(\theta), \theta)\right) dF(\theta) \equiv \int_{\underline{\theta}}^{\overline{\theta}} V^{**}(\theta) dF(\theta).$$

($\leq$) (i) The budget constraint in (*) is binding. If the budget constraint is not binding, increase c for some set of positive measure. This does not affect the other constraints and increases the objective. (ii) The policy function c in (*) is monotone increasing in θ. For suppose it is not, so that $c(s) = c_1 > c(\theta) = c_2$, for some $s < \theta$, and this inequality holds in neighborhoods of positive and equal F-measure around s and θ. Then simply interchange $c(s)$ and $c(\theta)$; this does not affect the budget constraint, but increases the objective, since $sc_2 + \theta c_1 > sc_1 + \theta c_2$. (iii) It is suboptimal for neither of the constraints $c \ge 0$ and $B' \ge \underline{B}$ to hold for the same θ. By increasing c and decreasing B' or vice versa, the objective can be increased.

These three steps show that it is suboptimal to violate the budget constraint for (**) for particular values of θ while satisfying it on average (*). Averaging of the budget sets of (**) therefore does not enlarge the set of feasible policies except on sets of F-measure zero. Form a sequence of feasible policies for problem (*), including the optimal policy. V^* is the supremum of averages of these policies, while $E(V^{**})$ is the average of suprema. Therefore by Fatou's Lemma (Taylor, 1973, p. 121),

$$V^* \le \int_{\underline{\theta}}^{\overline{\theta}} V^{**}(\theta) dF(\theta).$$

completing the argument. ∎

F. CONVERGENCE OF Ψ

If the initial distribution of credit or real balances did not match the stationary distribution Ψ, would an arbitrary distribution converge to it? This question is potentially very difficult because if the initial distribution were not equal to the stationary distribution, the interest rate might potentially be different from the stationary interest rate, and nonconstant as the distribution Ψ evolved. Because the threshold θ^* is independent of wealth, the question of the stationarity of the interest rate can be decoupled from that of the stationarity of Ψ. I approach the question by noting that if convergence can be demonstrated for the corresponding value states in the grandparents' contract, then convergence will occur for the corresponding distribution of real balances or credit with a constant rate of return.

LEMMA F.1 *Let $\beta\alpha^{-1} < 1$. Then the stochastic process of value states satisfies Condition M [Stokey and Lucas, 1989, p.348].*

PROOF: Since $\beta\alpha^{-1} < 1$, $\theta^* < \overline{\theta}$, which in turn means that $0 < F(\theta^*) < 1$. The result then follows from the fact that families inevitably transit to the minimum value state, $\underline{V}$, because they eventually become desperate. More formally, consider a set A that is a subset of the reals in excess of the minimum value state: $A \subset S \equiv \{x : x \geq \underline{V}\}$, where S is the set of all admissible value states. There are two cases. (i) If $\underline{V} \in A$, then for $s \in S$, $P(s, A)$ (the one-period transition probability from s to A) is $1 - F(\theta^*)$. (ii) If $\underline{V} \notin A$, then $P(s, A^c) \geq 1 - F(\theta^*)$ since $\underline{V} \subset A^c$. Thus it is necessary only to set $\epsilon = 1 - F(\theta^*)$. ■

THEOREM F.2 *(i) If the interest rate is a constant, any initial distribution function of credit, Ψ_k, converges strongly to the unique stationary Ψ. (ii) The rate of return is unaffected by the initial distribution of states.*

PROOF: (i) This follows from identifying each credit level with a value state as in section 10 of the main text. Then apply lemma F.1 and Theorem 11.12 of Stokey and Lucas (1989). (ii) Since $r = \alpha^{-1}$, a constant, and convergence holds for fixed α, the result follows. ■

I direct the reader's attention to appendix G which explicitly treats a special case.

G. NONSTATIONARY EQUILIBRIUM

It is instructive to examine the case that divides the stationary credit equilibria from the cases where an equilibrium does not exist. The dividing line case has the interesting property that it is *ex-ante* efficient, but the equilibrium distribution function of credit, Ψ, is not stationary. It suggests that optimality and stationarity may be incompatible in an equilibrium, and suggests that efficiency be studied as the limiting case of stationary but inefficient equilibria.

If the usual borrowing constraint, $B \geq -y/(r-1)$, were used (r is the gross interest rate), borrowers would remain permanently in debt, paying all endowment in interest. The class of borrowers would grow to include the entire population, and the class of creditors would continually shrink toward zero, with the wealth of each creditor tending to infinity. The equilibrium interest rate in this case is the same as the limiting gross interest rate of $1/F(\beta\overline{A})$. Suppose the economy begins with zero credit for all families. The fraction $F(\beta\overline{A})$ will be nondesperate and will save, and a fraction $(1 - F(\beta\overline{A}))$ will be desperate, borrowing $y/(r-1)$. There must be no aggregate debt in equilibrium, so the condition

$$(1 - F(\beta\overline{A}))(-\frac{y}{r-1}) + F(\beta\overline{A})y = 0$$

must hold. In the next period the initial debtors remain debtors, but their ranks are swelled by the fraction $(1 - F(\beta\overline{A}))$ of creditors who become desperate. Nondesperate creditors see the value of their savings grow by the interest rate, and in addition invest their endowment anew. Thus in the second period the condition

$$\left((1 - F(\beta\overline{A})) + (1 - F(\beta\overline{A}))F(\beta\overline{A})\right)(-\frac{y}{r-1}) + (F(\beta\overline{A}))^2\,(y + ry) = 0$$

must hold. Continuing this reasoning, after T periods,

$$(1 - F)\frac{1 - (F)^T}{1 - F}(-\frac{y}{r-1}) = (F)^T\frac{1 - r^T}{1 - r}$$

must hold, and it is algebraically straightforward to demonstrate that $r = 1/F(\beta\overline{A})$ solves this equation for all finite T.

These results translate directly into the contract and currency equilibria by the equivalence demonstrated in the main text.

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